

Risk contagion under regular variation and asymptotic tail independence

BIKRAMJIT DAS* and VICKY FASEN

Singapore University of Technology and Design
 8 Somapah Road, Singapore 487372
 E-mail: bikram@sutd.edu.sg

Karlsruhe Institute of Technology
 Englerstraße 2, 76131 Karlsruhe
 E-mail: vicky.fasen@kit.edu

Risk contagion concerns any entity dealing with large scale risks. Suppose $\mathbf{Z} = (Z_1, Z_2)$ denotes a risk vector pertaining to two components in some system. A relevant measurement of risk contagion would be to quantify the amount of influence of high values of Z_2 on Z_1 . This can be measured in a variety of ways. In this paper, we study two such measures: the quantity $\mathbb{E}[(Z_1 - t)_+ | Z_2 > t]$ called *Marginal Mean Excess* (MME) as well as the related quantity $\mathbb{E}[Z_1 | Z_2 > t]$ called *Marginal Expected Shortfall* (MES). Both quantities are indicators of risk contagion and useful in various applications ranging from finance, insurance and systemic risk to environmental and climate risk. We work under the assumptions of multivariate regular variation, hidden regular variation and asymptotic tail independence for the risk vector \mathbf{Z} . Many broad and useful model classes satisfy these assumptions. We present several examples and derive the asymptotic behavior of both MME and MES as the threshold $t \rightarrow \infty$. We observe that although we assume asymptotic tail independence in the models, MME and MES converge to ∞ under very general conditions; this reflects that the underlying weak dependence in the model still remains significant. Besides the consistency of the empirical estimators we introduce an extrapolation method based on extreme value theory to estimate both MME and MES for high thresholds t where little data are available. We show that these estimators are consistent and illustrate our methodology in both simulated and real data sets.

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1. Introduction

The presence of heavy-tail phenomena in data arising from a broad range of applications spanning from hydrology (Anderson and Meerschaert, 1998), finance (Smith, 2003), insurance (Embrechts, Klüppelberg and Mikosch, 1997), internet traffic (Crovello, Bestavros and Taqqu, 1999), social networks and random graphs (Durrett, 2010; Bollobás et al., 2003) and to risk management (Das, Embrechts and Fasen, 2013; Ibragimov, Jaffee and Walden, 2011) is well-documented; for further details see Resnick (2007), Nair, Wierman and Zwart (2016). Since heavy-tailed distributions often admit to non-existence of some higher order moments, measuring and assessing dependence in jointly heavy-tailed random variables poses a few challenges. Furthermore, one often encounters the phenomenon of *asymptotic tail independence* in the upper tails; which means given two jointly distributed heavy-tailed random variables, joint occurrence of very high (positive) values is extremely unlikely.

In this paper, we look at heavy-tailed random variables under the paradigm of *multivariate regular variation* possessing asymptotic tail independence in the upper tails and we study the average behavior of one of the variables given that the other one is large in an asymptotic sense. The presence of asymptotic tail independence might intuitively indicate that high values of one variable will have little influence on the

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expected behavior of the other; we observe that such a behavior is not always true. In fact under quite a general set of conditions we are able to calculate the asymptotic behavior of the expected value of a variable given that the other one is high.

One of the major applications of assessing such behavior is in terms of computing systemic risk, where one wants to assess risk contagion among two risk factors in a system. Proper quantification of systemic risk has been a topic of active research in the past few years; see [Adrian and Brunnermeier \(2011\)](#), [Biagini et al. \(2015\)](#), [Eisenberg and Noe \(2001\)](#), [Feinstein, Rudloff and Weber \(2015\)](#), [Brunnermeier and Cheridito \(2014\)](#), [Mainik and Schaanning \(2014\)](#) for further details. Our study concentrates on two such measures of risk in a bivariate set-up where both factors are heavy-tailed and possesses asymptotic tail independence.

Suppose $\mathbf{Z} = (Z_1, Z_2)$ denotes risk pertaining to components of a system. We study the behavior of two related quantities that captures the expected behavior of one risk, given that the other risk is high. First recall that for a random variable X and $0 < u < 1$ the Value-at-Risk (VaR) at level u is the quantile function

$$\text{VaR}_u(X) := \inf\{x \in \mathbb{R} : P(X > x) \leq 1 - u\} = \inf\{x \in \mathbb{R} : P(X \leq x) \geq u\}.$$

Definition 1.1 (MARGINAL MEAN EXCESS) For a random vector $\mathbf{Z} = (Z_1, Z_2)$ with $\mathbb{E}|Z_1| < \infty$ the *Marginal Mean Excess* (MME) at level p where $0 < p < 1$ is defined as:

$$\text{MME}(p) = \mathbb{E} \left[(Z_1 - \text{VaR}_{1-p}(Z_2))_+ | Z_2 > \text{VaR}_{1-p}(Z_2) \right]. \quad (1.1)$$

We interpret the MME as the expected excess of one risk Z_1 over the Value-at-Risk of Z_2 at level $(1 - p)$ given that the value of Z_2 is already greater than the same Value-at-Risk.

Definition 1.2 (MARGINAL EXPECTED SHORTFALL) For a random vector $\mathbf{Z} = (Z_1, Z_2)$ with $\mathbb{E}|Z_1| < \infty$ the *Marginal Expected Shortfall* (MES) at level p where $0 < p < 1$ is defined as:

$$\text{MES}(p) = \mathbb{E} [Z_1 | Z_2 > \text{VaR}_{1-p}(Z_2)]. \quad (1.2)$$

We interpret the MES as the expected shortfall of one risk given that the other risk is higher than its Value-at-risk at level $(1 - p)$. Note that smaller values of p lead to higher values of VaR_{1-p} .

In the context of systemic risk we might think of the conditioning variable Z_2 to be the risk of the entire system (for example, the entire market) and Z_1 as one component of the risk (for example, one financial institution). Hence, we are interested in the average or expected behavior of one specific component when the entire system is in distress. Although the problem is set up in a systemic risk context, the asymptotic behaviors of MME and MES are of interest in scenarios of risk contagion in a variety of disciplines.

Clearly, we are interested in computing both $\text{MME}(p)$ and $\text{MES}(p)$ for small values of p , which translates to Z_2 being over a high threshold t . In other words we are interested in estimators of $\mathbb{E}[(Z_1 - t)_+ | Z_2 > t]$ (for the MME) and $\mathbb{E}[Z_1 | Z_2 > t]$ (for the MES) for large values of t . An estimator for $\text{MES}(p)$ has been proposed by [Cai et al. \(2015\)](#) which is based on the asymptotic behavior of $\text{MES}(p)$; if $Z_1 \sim F_1$ and $Z_2 \sim F_2$, define

$$R(x, y) := \lim_{t \rightarrow \infty} t P \left(1 - F_1(Z_1) \leq \frac{x}{t}, 1 - F_2(Z_2) \leq \frac{y}{t} \right) \quad (1.3)$$

for $(x, y) \in [0, \infty)^2$. It is shown in [Cai et al. \(2015\)](#) that

$$\lim_{p \rightarrow 0} \frac{1}{\text{VaR}_{1-p}(Z_1)} \text{MES}(p) = \int_0^\infty R(x^{-\alpha_1}, 1) dx \quad (1.4)$$

if Z_1 has a regularly varying tail with tail parameter α_1 . In [Joe and Li \(2011\)](#) a similar result is presented under the further assumption of multivariate regular variation of the vector $\mathbf{Z} = (Z_1, Z_2)$; see [Zhu and Li \(2012\)](#), [Hua and Joe \(2012\)](#) as well in this context. Under the same assumptions, we can check that

$$\lim_{p \rightarrow 0} \frac{1}{\text{VaR}_{1-p}(Z_1)} \text{MME}(p) = \int_c^\infty R(x^{-\alpha_1}, 1) dx \quad (1.5)$$

where $c = \lim_{p \rightarrow 0} \frac{\text{VaR}_{1-p}(Z_2)}{\text{VaR}_{1-p}(Z_1)}$, if c exists and is finite. For c to be finite we require that the right-tail behavior of Z_2 be equivalent to that of Z_1 ($c > 0$) or lighter than that of Z_1 ($c = 0$). Note that in both (1.4) and (1.5), the rate of increase of the risk measure is determined by the tail behavior of Z_1 ; the tail behavior of Z_2 has no apparent influence. However, these results make sense only when the right hand sides of (1.4) and (1.5) are both non-zero and finite. Thus, we obtain that as $p \downarrow 0$,

$$\text{MME}(p) \sim \text{const. VaR}_{1-p}(Z_1), \quad \text{and} \quad \text{MES}(p) \sim \text{const. VaR}_{1-p}(Z_1).$$

Unfortunately, if Z_1, Z_2 are asymptotic upper tail *independent* then $R(x, y) \equiv 0$ (cf. Remark 2.3 below) which implies that the limits in (1.4) and (1.5), are both 0 as well and hence, are not that useful.

Consequently, the results in Cai et al. (2015) make sense only if the random vector \mathbf{Z} has positive upper tail dependence which means that Z_1 and Z_2 can take high values together with a positive probability; examples of multivariate regularly varying random vectors producing such strong dependence can be found in Hua and Joe (2014). A classical example for asymptotic tail independence, especially in financial risk modeling, is when risk factors Z_1 and Z_2 are both Pareto-tailed with a Gaussian copula and any correlation $\rho < 1$ (Das, Embrechts and Fasen, 2013); this model has asymptotic upper tail independence leading to $R \equiv 0$. The result in (1.4) and (1.5), and hence, in Cai et al. (2015) would provide a null estimate which is not very informative. Hence, in such a case one might be inclined to believe that $\mathbb{E}(Z_1|Z_2 > t) \sim \mathbb{E}(Z_1)$ and $\mathbb{E}((Z_1 - t)_+|Z_2 > t) \sim 0$ as Z_1 and Z_2 are asymptotically tail independent. However, we will see that depending on the Gaussian copula parameter ρ we might even have $\lim_{t \rightarrow \infty} \mathbb{E}((Z_1 - t)_+|Z_2 > t) = \infty$. Hence, in this case it would be nice if we could find the right rate of convergence of $\text{MME}(p)$ to a non-zero constant.

In this paper we investigate the asymptotic behavior of $\text{MME}(p)$ and $\text{MES}(p)$ as $p \downarrow 0$ under the assumption of regular variation and hidden regular variation of the risk vector \mathbf{Z} exhibiting asymptotic tail independence. We will see that for a very general class of models $\text{MME}(p)$ and $\text{MES}(p)$, respectively behave like a regularly varying function with negative index for $p \downarrow 0$, and hence, converge to ∞ although the tails are asymptotically tail independent. However, the rate of convergence is slower than in the asymptotically tail dependent case as presented in Cai et al. (2015). This result is an interplay between the tail behavior and the strength of dependence of the two variables in the tails. The behavior of MES in the asymptotically tail independent case has been addressed to some extent in (Hua and Joe, 2014, Section 3.4) for certain copula structures with Pareto margins. We address the asymptotically tail independent case in further generality. For the MME , we can provide results with fewer technical assumptions than for the case of MES and hence, we cover a broader class of asymptotically tail independent models. The knowledge of the asymptotic behavior of the MME and the MES helps us in proving consistency of their empirical estimators. However, in a situation where data are scarce or even unavailable in the tail region of interest, an empirical estimator is clearly unsuitable. Hence, we also provide consistent estimators using methods from extreme value theory which work when data availability is limited in the tail regions.

The paper is structured as follows: In Section 2 we briefly discuss the notion of multivariate and hidden regular variation. We also list a set of assumptions that we impose on our models in order to obtain limits of the quantities MME and MES under appropriate scaling. We illustrate a few examples which satisfy the assumptions of our models including the Bernoulli mixture model and additive models for generating hidden regular variation. The main results of the paper regarding the asymptotic behavior of the MME and the MES are discussed in Section 3. Estimation methods for the risk measures MME and MES are provided in Section 4. Consistency of the empirical estimators are the topic of Section 4.1, whereas, we present consistent estimators based on methods from extreme value theory in Section 4.2. Finally, we validate our method on real and simulated data in Section 5 with brief concluding remarks in Section 6.

In the following we denote by \xrightarrow{v} vague convergence of measures, by \Rightarrow weak convergence of measures and by \xrightarrow{P} convergence in probability.

2. Preliminaries

For this paper we restrict our attention to non-negative random variables in a bivariate setting. We discuss multivariate and hidden regular variation in Section 2.1. A few technical assumptions that we use throughout the paper are listed in Section 2.2. A selection of model examples that satisfy our assumptions is relegated to Section 2.3.

2.1. Regular variation

First, recall that a measurable function $f : (0, \infty) \rightarrow (0, \infty)$ is *regularly varying* at ∞ with index $\rho \in \mathbb{R}$ if

$$\lim_{t \rightarrow \infty} \frac{f(tx)}{f(t)} = x^\rho$$

for any $x > 0$ and we write $f \in \mathcal{RV}_\rho$. If the index of regular variation is 0 we call the function slowly varying as well. Note that in contrast, we say f is regularly varying at 0 with index ρ if $\lim_{t \rightarrow 0} f(tx)/f(t) = x^\rho$ for any $x > 0$. In this paper, unless otherwise specified, regular variation means regular variation at infinity. A random variable X with distribution function F has a regularly varying tail if $\bar{F} = 1 - F \in \mathcal{RV}_{-\alpha}$ for some $\alpha \geq 0$. We often write $X \in \mathcal{RV}_{-\alpha}$ by abuse of notation.

We use the notion of \mathbb{M} -convergence to define regular variation in more than one dimension; for further details see [Lindskog, Resnick and Roy \(2014\)](#), [Hult and Lindskog \(2006\)](#), [Das, Mitra and Resnick \(2013\)](#). We restrict to two dimensions here since we deal with bivariate distributions in this paper, although the definitions provided hold in general for any finite dimension. Suppose $\mathbb{C}_0 \subset \mathbb{C} \subset [0, \infty)^2$ where \mathbb{C}_0 and \mathbb{C} are closed cones containing $\{(0, 0)\} \in \mathbb{R}^2$. By $\mathbb{M}(\mathbb{C} \setminus \mathbb{C}_0)$ we denote the class of Borel measures on $\mathbb{C} \setminus \mathbb{C}_0$ which are finite on subsets bounded away from \mathbb{C}_0 . Then $\mu_n \xrightarrow{\mathbb{M}} \mu$ in $\mathbb{M}(\mathbb{C} \setminus \mathbb{C}_0)$ if $\mu_n(f) \rightarrow \mu(f)$ for all continuous and bounded functions on $\mathbb{C} \setminus \mathbb{C}_0$ whose supports are bounded away from \mathbb{C}_0 .

Definition 2.1 (MULTIVARIATE REGULAR VARIATION) A random vector $\mathbf{Z} = (Z_1, Z_2) \in \mathbb{C}$ is (*multivariate*) *regularly varying* on $\mathbb{C} \setminus \mathbb{C}_0$, if there exist a function $b(t) \uparrow \infty$ and a non-zero measure $\nu(\cdot) \in \mathbb{M}(\mathbb{C} \setminus \mathbb{C}_0)$ such that as $t \rightarrow \infty$,

$$\nu_t(\cdot) := tP(\mathbf{Z}/b(t) \in \cdot) \xrightarrow{\mathbb{M}} \nu(\cdot) \quad \text{in } \mathbb{M}(\mathbb{C} \setminus \mathbb{C}_0). \quad (2.1)$$

Moreover, we can check that the limit measure has the homogeneity property: $\nu(cA) = c^{-\alpha}\nu(A)$ for some $\alpha > 0$. We write $\mathbf{Z} \in \mathcal{MRV}(\alpha, b, \nu, \mathbb{C} \setminus \mathbb{C}_0)$ and sometimes write MRV for multivariate regular variation.

In the first stage, multivariate regular variation is defined on the space $\mathbb{E} = [0, \infty)^2 \setminus \{(0, 0)\} = \mathbb{C} \setminus \mathbb{C}_0$ where $\mathbb{C} = [0, \infty)^2$ and $\mathbb{C}_0 = \{(0, 0)\}$. But sometimes we need to define further regular variation on subspaces of \mathbb{E} , since the limit measure ν as obtained in (2.1) turns out to be concentrated on a subspace of \mathbb{E} . The most likely way this happens is through *asymptotic tail independence* of random variables.

Definition 2.2 (ASYMPTOTIC TAIL INDEPENDENCE) A random vector $\mathbf{Z} = (Z_1, Z_2) \in [0, \infty)^2$ is called *asymptotically independent (in the upper tail)* if

$$\lim_{p \rightarrow 0} P(Z_2 > F_{Z_2}^{\leftarrow}(1-p) | Z_1 > F_{Z_1}^{\leftarrow}(1-p)) = 0.$$

Independent random vectors are trivially asymptotically tail independent. Note that asymptotic upper tail independence of $\mathbf{Z} \in \mathcal{MRV}(\alpha, b, \nu, \mathbb{E})$ implies $\nu((0, \infty) \times (0, \infty)) = 0$ for the limit measure ν . On the other hand, for the converse proposition, if Z_1 and Z_2 are both marginally regularly varying in the right tail with $\lim_{t \rightarrow \infty} P(Z_1 > t)/P(Z_2 > t) = 1$, then $\nu((0, \infty) \times (0, \infty)) = 0$ implies asymptotic upper tail independence as well (cf. [Resnick, 1987](#), Proposition 5.27). However, this implication does not hold true in general, e.g., for some regularly varying random variable $X \in \mathcal{RV}_{-\alpha}$ the random vector (X, X^2) is multivariate regularly varying with limit measure $\nu((0, \infty) \times (0, \infty)) = 0$; but of course (X, X^2) is asymptotically tail-dependent.

Remark 2.3 Assume (w.l.o.g.) that F_1, F_2 are strictly increasing continuous distribution functions with unique survival copula \widehat{C} (cf. [Nelsen \(2006\)](#)) such that

$$P(Z_1 > x, Z_2 > y) = \widehat{C}(\overline{F}_1(x), \overline{F}_2(y)) \quad \text{for } (x, y) \in \mathbb{R}^2.$$

Now asymptotic upper tail independence of (Z_1, Z_2) implies that

$$\begin{aligned} R(x, y) &= \lim_{t \rightarrow \infty} t P(1 - F_1(Z_1) \leq x/t, 1 - F_2(Z_2) \leq y/t) \\ &= \lim_{t \rightarrow \infty} t \widehat{C}\left(\frac{x}{t}, \frac{y}{t}\right) \leq \max(x, y) \lim_{s \rightarrow 0} \frac{\widehat{C}(s, s)}{s} = 0. \end{aligned}$$

Hence, the estimator presented in [Cai et al. \(2015\)](#) for MES provides a trivial estimator in this setting.

Consequently, in the asymptotically tail independent case where the tails are equivalent we would approximate the joint tail probability by $P(Z_2 > x | Z_1 > x) \approx 0$ for large thresholds x and conclude that risk contagion between Z_1 and Z_2 is absent. This conclusion may be naive; hence the notion of *hidden regular variation* on $\mathbb{E}_0 = [0, \infty)^2 \setminus (\{0\} \times [0, \infty) \cup [0, \infty) \times \{0\}) = (0, \infty)^2$ was introduced in [Resnick \(2002\)](#). Note that we do not assume that the marginal tails of \mathbf{Z} are necessarily equivalent in order to define hidden regular variation, which is usually done in [Resnick \(2002\)](#).

Definition 2.4 (HIDDEN REGULAR VARIATION) A regularly varying random vector \mathbf{Z} on \mathbb{E} possesses *hidden regular variation* on $\mathbb{E}_0 = (0, \infty)^2$ with index $\alpha_0 (\geq \alpha > 0)$ if there exist scaling functions $b(t) \in \mathcal{RV}_{1/\alpha}$ and $b_0(t) \in \mathcal{RV}_{1/\alpha_0}$ with $b(t)/b_0(t) \rightarrow \infty$ and limit measures ν, ν_0 such that

$$\mathbf{Z} \in \mathcal{MRV}(\alpha, b, \nu, \mathbb{E}) \cap \mathcal{MRV}(\alpha_0, b_0, \nu_0, \mathbb{E}_0).$$

We write $\mathbf{Z} \in \mathcal{HRV}(\alpha_0, b_0, \nu_0)$ and sometimes write HRV for hidden regular variation.

For example, say Z_1, Z_2 are iid random variables with distribution function $F(x) = 1 - x^{-1}, x > 1$. Here $\mathbf{Z} = (Z_1, Z_2)$ possesses MRV on \mathbb{E} , asymptotic tail independence and HRV on \mathbb{E}_0 . Specifically, $\mathbf{Z} \in \mathcal{MRV}(\alpha = 1, b(t) = t, \nu, \mathbb{E}) \cap \mathcal{MRV}(\alpha_0 = 2, b_0(t) = \sqrt{t}, \nu_0, \mathbb{E}_0)$ where for $x > 0, y > 0$,

$$\nu([(0, 0), (x, y)]^c) = \frac{1}{x} + \frac{1}{y} \quad \text{and} \quad \nu_0([x, \infty) \times [y, \infty)) = \frac{1}{xy}.$$

Lemma 2.5. $\mathbf{Z} \in \mathcal{MRV}(\alpha, b, \nu, \mathbb{E}) \cap \mathcal{HRV}(\alpha_0, b_0, \nu_0, \mathbb{E}_0)$ implies that \mathbf{Z} is asymptotically tail independent.

Proof. Let $b_i(t) = (1/(1 - F_i))^\leftarrow(t)$ where $Z_i \sim F_i, i = 1, 2$. Due to the assumptions we have

$$\lim_{t \rightarrow \infty} \frac{\max(b_1(t), b_2(t))}{b_0(t)} = \infty \quad \text{and} \quad \liminf_{t \rightarrow \infty} \frac{\min(b_1(t), b_2(t))}{b_0(t)} \geq 1.$$

Without loss of generality $b_1(t)/b_0(t) \rightarrow \infty$. Then for any $M > 0$ there exists a $t_0 = t_0(M)$ so that $b_1(t) \geq Mb_0(t)$ for any $t \geq t_0$. Hence, for $x, y > 0$

$$\begin{aligned} \lim_{t \rightarrow \infty} t P(1 - F_1(Z_1) \leq x/t, 1 - F_2(Z_2) \leq y/t) &= \lim_{t \rightarrow \infty} t P(Z_1 \geq b_1(t/x), Z_2 \geq b_2(t/y)) \\ &\leq \lim_{t \rightarrow \infty} t P(Z_1 \geq Mb_0(t/x), Z_2 \geq 2^{-1}b_0(t/y)) \\ &\leq C(x, y) \nu_0([M, \infty) \times [2^{-1}, \infty)) \xrightarrow{M \rightarrow \infty} 0, \end{aligned}$$

so that \mathbf{Z} is asymptotically tail independent. □

Remark 2.6 The assumption $\mathbf{Z} \in \mathcal{MRV}(\alpha, b, \nu, \mathbb{E}) \cap \mathcal{MRV}(\alpha_0, b_0, \nu_0, \mathbb{E}_0)$ and \mathbf{Z} is asymptotic upper tail independent already implies that $\mathbf{Z} \in \mathcal{HRV}(\alpha_0, b_0, \nu_0)$ ([Resnick, 2002](#); [Maulik and Resnick, 2005](#)). Consequently $\lim_{t \rightarrow \infty} b(t)/b_0(t) = \infty$ as well.

2.2. Assumptions

In this section we list assumptions on the random variables for which we show consistency of relevant estimators in the paper. Parts of the assumptions are to fix notations for future results. Let $\mathbf{Z} = (Z_1, Z_2) \in [0, \infty)^2$.

Assumption A

(A1) Assume $\mathbf{Z} \in \mathcal{MRV}(\alpha, b, \nu, \mathbb{E})$ where

$$b(t) = (1/P(\max(Z_1, Z_2) > \cdot))^{\leftarrow}(t) = \overline{F}_{\max(Z_1, Z_2)}^{\leftarrow}(1/t) \in \mathcal{RV}_{1/\alpha}.$$

(A2) $\mathbb{E}|Z_1| < \infty$.

(A3) $b_2(t) := \overline{F}_{Z_2}^{\leftarrow}(1/t)$ for $t \geq 0$.

(A4) Without loss of generality we assume that the support of Z_1 is $[1, \infty)$. A constant shift would not affect the tail properties of MME or MES.

(A5) Moreover, assume that $\mathbf{Z} \in \mathcal{MRV}(\alpha_0, b_0, \nu_0, \mathbb{E}_0)$ with $\alpha_0 \geq \alpha \geq 1$, where

$$b_0(t) = (1/P(\min(Z_1, Z_2) > \cdot))^{\leftarrow}(t) = \overline{F}_{\min(Z_1, Z_2)}^{\leftarrow}(1/t) \in \mathcal{RV}_{1/\alpha_0},$$

and $b(t)/b_0(t) \rightarrow \infty$.

Lemma 2.7. *Let $\overline{F}_{Z_2} \in \mathcal{RV}_{-\beta}$, $\beta > 0$. Then Assumption A implies $\alpha \leq \beta \leq \alpha_0$.*

Proof. First of all, $\beta \geq \alpha$ since otherwise $\mathbf{Z} \in \mathcal{MRV}(\alpha, b, \nu, \mathbb{E})$ cannot hold. Moreover,

$$1 \sim tP(Z_1 > b_0(t), Z_2 > b_0(t)) \leq tP(Z_2 > b_0(t)) \in \mathcal{RV}_{1-\frac{\beta}{\alpha_0}}. \quad (2.2)$$

Thus, if $\alpha_0 < \beta$ then $\lim_{t \rightarrow \infty} tP(Z_2 > b_0(t)) = 0$ which is a contradiction to (2.2). \square

Remark 2.8 In general, we see from this that under Assumption A, $\liminf_{t \rightarrow \infty} tP(Z_2 > b_0(t)) \geq 1$ and hence, for any $\epsilon > 0$ there exist $C_1(\epsilon) > 0$, $C_2(\epsilon) > 0$ and $x_0(\epsilon) > 0$ such that

$$C_1(\epsilon)x^{-\alpha_0-\epsilon} \leq P(Z_2 > x) \leq C_2(\epsilon)x^{-\alpha+\epsilon}$$

for any $x \geq x_0(\epsilon)$.

We need a couple of more conditions, especially on the joint tail behavior of $\mathbf{Z} = (Z_1, Z_2)$ in order to talk about the limit behavior of MME(p) and MES(p) as $p \downarrow 0$. We impose the following assumptions on the distribution of \mathbf{Z} . Assumption (B1) is imposed to find the limit of MME in (1.1) whereas both (B1) and (B2) (which are clubbed together as Assumption B) are imposed to find the limit in (1.2), of course, both under appropriate scaling.

Assumption B

$$(B1) \quad \lim_{M \rightarrow \infty} \lim_{t \rightarrow \infty} \int_M^\infty \frac{P(Z_1 > xt, Z_2 > t)}{P(Z_1 > t, Z_2 > t)} dx = 0.$$

$$(B2) \quad \lim_{M \rightarrow \infty} \lim_{t \rightarrow \infty} \int_0^{1/M} \frac{P(Z_1 > xt, Z_2 > t)}{P(Z_1 > t, Z_2 > t)} dx = 0.$$

Assumption (B1) and Assumption (B2) deal with tail integrability near infinity and near zero for a specific integrand, respectively that comes up in calculating limits of MME and MES. The following trivially provides a sufficient condition for (B1).

Lemma 2.9. *If there exists an integrable function $g : [0, \infty) \rightarrow [0, \infty)$ with*

$$\sup_{t \geq t_0} \frac{P(Z_1 > y, Z_2 > t)}{tP(Z_1 > t, Z_2 > t)} \leq g(y)$$

for $y > 0$ and some $t_0 > 0$ then (B1) is satisfied.

Lemma 2.10. *Let Assumption A hold.*

(a) *Then (B2) implies*

$$\lim_{t \rightarrow \infty} \frac{P(Z_2 > t)}{tP(Z_1 > t, Z_2 > t)} = 0. \quad (2.3)$$

(b) *Suppose $\bar{F}_{Z_2} \in \mathcal{RV}_{-\beta}$ with $\alpha \leq \beta \leq \alpha_0$. Then $\alpha_0 \leq \beta + 1$ is a necessary and $\alpha_0 < \beta + 1$ is a sufficient condition for (2.3) to hold. Hence, $\alpha_0 \leq \beta + 1$ is a necessary condition for Assumption (B2) as well.*

Proof.

(a) Since the support of Z_1 is $[1, \infty)$ we get for large t

$$\frac{P(Z_2 > t)}{tP(Z_1 > t, Z_2 > t)} = \int_0^{1/t} \frac{P(Z_1 > xt, Z_2 > t)}{P(Z_1 > t, Z_2 > t)} dx \leq \int_0^{1/M} \frac{P(Z_1 > xt, Z_2 > t)}{P(Z_1 > t, Z_2 > t)} dx \xrightarrow{t, M \rightarrow \infty} 0.$$

But the left hand side is independent of M so that the claim follows.

(b) In this case $\frac{P(Z_2 > t)}{tP(Z_1 > t, Z_2 > t)} \in \mathcal{RV}_{-\beta-1+\alpha_0}$ from which the statement follows. \square

Remark 2.11 If Z_1, Z_2 are independent then under the assumptions of Lemma 2.10(b), $\alpha_0 = \alpha + \beta$. Moreover if $1 < \alpha \leq \beta$ then clearly $\alpha_0 = \alpha + \beta > 1 + \beta$ and $\alpha \leq 1 + \beta$ cannot hold. Hence, Assumption (B2) is not valid if Z_1 and Z_2 are independent. In other words, Assumption (B2) signifies that although Z_1, Z_2 are asymptotically upper tail independent, there is an underlying dependence between Z_1 and Z_2 which is absent in the independent case.

2.3. A selection of models

In this section we study a few models which provide sufficient conditions such that A and B hold. Further examples, in particular including copula examples can be found in Das and Fasen (2016).

2.3.1. Mixture representation

First we look at models that are generated in an additive fashion (Weller and Cooley, 2014; Das and Resnick, 2015). We will observe that many models can be generated using the additive technique. The dependence properties assumed here are satisfied by many copulas used in practice. This will be further exemplified.

Model C Suppose $\mathbf{Z} = (Z_1, Z_2), \mathbf{Y} = (Y_1, Y_2), \mathbf{V} = (V_1, V_2)$ are random vectors in $[0, \infty)^2$ such that $\mathbf{Z} = \mathbf{Y} + \mathbf{V}$. Assume the following holds:

- (C1) $\mathbf{Y} \in \mathcal{MRV}(\alpha, b, \nu, \mathbb{E})$ where $\alpha \geq 1$.
- (C2) Y_1, Y_2 are independent random variables.
- (C3) $\bar{F}_{Y_2} \in \mathcal{RV}_{-\alpha^*}$, $1 \leq \alpha \leq \alpha^*$.
- (C4) $\mathbf{V} \in \mathcal{MRV}(\alpha_0, b_0, \nu_0, \mathbb{E})$ and does not possess asymptotic tail independence where $\alpha \leq \alpha_0$ and

$$\lim_{t \rightarrow \infty} \frac{P(\|\mathbf{V}\| > t)}{P(\|\mathbf{Y}\| > t)} = 0.$$

(C5) \mathbf{Y} and \mathbf{V} are independent.

(C6) $\alpha \leq \alpha_0 < 1 + \alpha^*$.

(C7) $\mathbb{E}|Z_1| < \infty$.

Of course, we would like to know, when Model C satisfy Assumptions A and B; moreover, when is $Z \in \mathcal{HRV}(\alpha_0, b_0, \nu_0, \mathbb{E}_0)$? The next theorem provides a general result to answer these questions in certain special cases.

Theorem 2.12. *Let $\mathbf{Z} = \mathbf{Y} + \mathbf{V}$ be as in Model C. Then the following statements hold:*

- (a) $\mathbf{Z} \in \mathcal{MRV}(\alpha, b, \nu, \mathbb{E}) \cap \mathcal{HRV}(\alpha_0, b_0, \nu_0, \mathbb{E}_0)$ and satisfies B.
(b) Suppose $Y_1 = 0$. Then $(Z_1, Z_1 + Z_2) \in \mathcal{MRV}(\alpha, b, \nu, \mathbb{E}) \cap \mathcal{HRV}(\alpha_0, b_0, \nu_0^+, \mathbb{E}_0)$ with

$$\nu_0^+(A) = \nu_0(\{(v_1, v_2) \in \mathbb{E}_0 : (v_1, v_1 + v_2) \in A\}) \quad \text{for } A \in \mathcal{B}(\mathbb{E}_0)$$

and satisfies B.

- (c) Suppose $\liminf_{t \rightarrow \infty} \frac{P(Y_1 > t)}{P(Y_2 > t)} > 0$. Then $(Z_1, \min(Z_1, Z_2)) \in \mathcal{MRV}(\alpha, b, \nu^{\min}, \mathbb{E}) \cap \mathcal{HRV}(\alpha_0, b_0, \nu_0^{\min}, \mathbb{E}_0)$ with

$$\begin{aligned} \nu^{\min}(A) &= \nu(\{(y_1, 0) \in \mathbb{E} : (y_1, 0) \in A\}) \quad \text{for } A \in \mathcal{B}(\mathbb{E}), \\ \nu_0^{\min}(A) &= \nu_0(\{(v_1, v_2) \in \mathbb{E}_0 : (v_1, \min(v_1, v_2)) \in A\}) \quad \text{for } A \in \mathcal{B}(\mathbb{E}_0) \end{aligned}$$

and satisfies B.

- (d) Suppose $Y_1 = 0$. Then $(Z_1, \max(Z_1, Z_2)) \in \mathcal{MRV}(\alpha, b, \nu, \mathbb{E}) \cap \mathcal{HRV}(\alpha_0, b_0, \nu_0^{\max}, \mathbb{E}_0)$ with

$$\nu_0^{\max}(A) = \nu_0(\{(v_1, v_2) \in \mathbb{E}_0 : (v_1, \max(v_1, v_2)) \in A\}) \quad \text{for } A \in \mathcal{B}(\mathbb{E}_0)$$

and satisfies B.

For a proof of this theorem we refer to Das and Fasen (2016).

Remark 2.13 Note that, in a systemic risk context where the entire system consists of two institutions with risks Z_1 and Z_2 , the above theorem addresses the variety of ways a systemic risk model can be constructed. If risk is just additive we could refer to part (b), if the system is at risk when both institutions are at risk then we can refer to part (c) and if the global risk is connected to any of the institutions being at risk then we can refer to the model in part (d). Hence, many kinds of models for calculating systemic risk can be obtained under such a model assumption.

2.3.2. Bernoulli model

Next we investigate an example generated by using a mixture method for getting hidden regular variation in a non-standard regularly varying model (Das, Mitra and Resnick, 2013).

Example 2.14 Suppose X_1, X_2, X_3 are independent Pareto random variables with parameters α, α_0 and γ , respectively, where $1 < \alpha < \alpha_0 < \gamma$ and $\alpha_0 < 1 + \alpha$. Let B be a Bernoulli(q) random variable independent of X_1, X_2, X_3 . Now define

$$\mathbf{Z} = (Z_1, Z_2) = B(X_1, X_3) + (1 - B)(X_2, X_2).$$

This is a popular example, see Resnick (2002); Maulik and Resnick (2005); Das and Resnick (2015). Note that

$$P(\max(Z_1, Z_2) > t) \sim qt^{-\alpha} \quad \text{and} \quad P(\min(Z_1, Z_2) > t) \sim P(Z_2 > t) \sim (1 - q)t^{-\alpha_0} \quad (t \rightarrow \infty),$$

so that $b(1/p) \sim q^{\frac{1}{\alpha}} p^{-\frac{1}{\alpha}}$, $b_0(1/p) \sim b_2(1/p) \sim (1 - q)^{\frac{1}{\alpha_0}} p^{-\frac{1}{\alpha_0}}$ as $p \downarrow 0$. The point measure at 0 is denoted by ϵ_0 and the limit measure on \mathbb{E} concentrates on the two axes. We will look at usual MRV which is given on \mathbb{E} by

$$tP\left(\left(\frac{Z_1}{b(t)}, \frac{Z_2}{b(t)}\right) \in dx dy\right) \xrightarrow{\mathbb{M}} \alpha x^{-\alpha-1} dx \cdot \epsilon_0(dy) =: \nu(dx dy) \quad (t \rightarrow \infty) \quad \text{in } \mathbb{M}(\mathbb{E}),$$

where the limit measure lies on the x-axis. Hence, we seek HRV in the next step on $\mathbb{E} \setminus \{x\text{-axis}\} = [0, \infty) \times (0, \infty)$ and get

$$tP\left(\left(\frac{Z_1}{b_0(t)}, \frac{Z_2}{b_0(t)}\right) \in dx dy\right) \xrightarrow{\mathbb{M}} \alpha_0 x^{-\alpha_0-1} dx \cdot \epsilon_x(dy) =: \nu_0(dx dy) \quad (t \rightarrow \infty) \quad \text{in } \mathbb{M}(\mathbb{E} \setminus \{x\text{-axis}\}).$$

As usual the point measure at x is denoted by ϵ_x and the limit measure lies on the diagonal where $x = y$. Thus, we have for any $x \geq 1$,

$$\nu_0((x, \infty) \times (1, \infty)) = x^{-\alpha_0}.$$

Now, we can explicitly calculate the values of MME and MES too. For $0 < p < 1$:

$$\begin{aligned} \text{MES}(p) &= \frac{1}{q\text{VaR}_{1-p}(Z_2)^{-\gamma} + (1-q)\text{VaR}_{1-p}(Z_2)^{-\alpha_0}} \left[\frac{q\alpha}{\alpha-1} \text{VaR}_{1-p}(Z_2)^{-\gamma} + \frac{(1-q)\alpha_0}{\alpha_0-1} \text{VaR}_{1-p}(Z_2)^{-\alpha_0+1} \right], \\ \text{MME}(p) &= \frac{1}{q\text{VaR}_{1-p}(Z_2)^{-\gamma} + (1-q)\text{VaR}_{1-p}(Z_2)^{-\alpha_0}} \left[\frac{q}{\alpha-1} \text{VaR}_{1-p}(Z_2)^{-\gamma-\alpha+1} + \frac{(1-q)}{\alpha_0-1} \text{VaR}_{1-p}(Z_2)^{-\alpha_0+1} \right]. \end{aligned}$$

Therefore,

$$\frac{pb_0^{\leftarrow}(b_2(1/p))}{b_2(1/p)} \text{MME}(p) \sim \frac{1}{b_2(1/p)} \text{MME}(p) \sim \frac{1}{\alpha_0-1} = \int_1^\infty \nu_0((x, \infty) \times (1, \infty)) dx \quad (p \downarrow 0),$$

and

$$\frac{pb_0^{\leftarrow}(b_2(1/p))}{b_2(1/p)} \text{MES}(p) \sim \frac{1}{b_2(1/p)} \text{MES}(p) \sim \frac{\alpha_0}{\alpha_0-1} = \int_0^\infty \nu_0((x, \infty) \times (1, \infty)) dx \quad (p \downarrow 0).$$

3. Asymptotic behavior of the MME and the MES

3.1. Asymptotic behavior of the MME

For asymptotically independent risks, from (1.5) and Remark 2.3 we have that

$$\lim_{p \rightarrow 0} \frac{1}{\text{VaR}_{1-p}(Z_1)} \text{MME}(p) = 0,$$

which doesn't provide us much in the way of identifying the rate of increase (or decrease) of $\text{MME}(p)$. The aim of this section is to get a version of (1.5) for the asymptotically tail independent case and obtain a limit behavior of $\text{MME}(p)$ which is presented in the next theorem.

Theorem 3.1. *Suppose $\mathbf{Z} = (Z_1, Z_2) \in [0, \infty)^2$ satisfies Assumption A and (B1). Then*

$$\lim_{p \downarrow 0} \frac{pb_0^{\leftarrow}(b_2(1/p))}{b_2(1/p)} \text{MME}(p) = \lim_{p \downarrow 0} \frac{pb_0^{\leftarrow}(\text{VaR}_{1-p}(Z_2))}{\text{VaR}_{1-p}(Z_2)} \text{MME}(p) = \int_1^\infty \nu_0((x, \infty) \times (1, \infty)) dx. \quad (3.1)$$

Moreover, $0 < \int_1^\infty \nu_0((x, \infty) \times (1, \infty)) dx < \infty$.

Proof. We know that for a non-negative random variable W , we have $\mathbb{E}W = \int_0^\infty P(W > x) dx$. Let $t = b_2(1/p)$. Also note that $b_0^{\leftarrow}(t) = 1/P(\min(Z_1, Z_2) > t) = 1/P(Z_1 > t, Z_2 > t)$. Then

$$\begin{aligned} \frac{pb_0^{\leftarrow}(b_2(1/p))}{b_2(1/p)} \text{MME}(p) &= \frac{\bar{F}_{Z_2}(t)b_0^{\leftarrow}(t)}{t} \mathbb{E}((Z_1 - t)_+ | Z_2 > t) \\ &= \frac{P(Z_2 > t)b_0^{\leftarrow}(t)}{t} \int_t^\infty \frac{P(Z_1 > x, Z_2 > t)}{P(Z_2 > t)} dx \\ &= \int_t^\infty \frac{P(Z_1 > x, Z_2 > t)}{tP(Z_1 > t, Z_2 > t)} dx \\ &= \left[\int_1^M + \int_M^\infty \right] \frac{P(Z_1 > tx, Z_2 > t)}{P(Z_1 > t, Z_2 > t)} dx. \end{aligned}$$

Define the function $f : \mathbb{E}_0 \rightarrow [0, M]$ as $f(z_1, z_2) = (\min(z_1, M) - 1)\mathbf{1}_{\{z_1 > 1, z_2 > 1\}}$ which is continuous, bounded and has compact support on \mathbb{E}_0 . Then with $\nu_{0,t}$ as defined in (2.1) we get

$$\begin{aligned} \int_1^M \frac{P(Z_1 > tx, Z_2 > t)}{P(Z_1 > t, Z_2 > t)} dx &= \int_{\mathbb{E}_0} f(z_1, z_2) \nu_{0, b_0^-(t)}(dz_1, dz_2) \\ &\xrightarrow{t \rightarrow \infty} \int_{\mathbb{E}_0} f(z_1, z_2) \nu_0(dz_1, dz_2) = \int_1^M \nu_0((x, \infty) \times (1, \infty)) dx, \end{aligned}$$

where we used \mathbb{M} -convergence as in (2.1). Since

$$\lim_{M \rightarrow \infty} \int_1^M \nu_0((x, \infty) \times (1, \infty)) dx = \int_1^\infty \nu_0((x, \infty) \times (1, \infty)) dx$$

and $\lim_{M \rightarrow \infty} \lim_{t \rightarrow \infty} \int_M^\infty \frac{P(Z_1 > tx, Z_2 > t)}{P(Z_1 > t, Z_2 > t)} dx = 0$ by Assumption (B1) the statement (3.1) follows. Moreover, $\nu_0((x, \infty) \times (x, \infty))$ is homogeneous of order $-\alpha_0$ so that

$$\int_1^\infty \nu_0((x, \infty) \times (1, \infty)) dx \geq \int_1^\infty \nu_0((x, \infty) \times (x, \infty)) dx = \nu_0((1, \infty) \times (1, \infty)) \int_1^\infty x^{-\alpha_0} dx > 0.$$

Finally, Assumption (B1) and the fact that $\int_1^M \nu_0((x, \infty) \times (1, \infty)) dx \leq M \nu_0((1, \infty) \times (1, \infty)) < \infty$ implies

$$\int_1^\infty \nu_0((1, \infty) \times (1, \infty)) dx < \infty.$$

□

Corollary 3.2. Suppose $\mathbf{Z} = (Z_1, Z_2)$ satisfies Assumptions A, (B1) and $\bar{F}_{Z_2} \in \mathcal{RV}_{-\beta}$ for some $\alpha \leq \beta \leq \alpha_0$. Then $MME(1/t) \in \mathcal{RV}_{\frac{1+\beta-\alpha_0}{\beta}}$. For $1 + \beta > \alpha_0$ we have $\lim_{p \rightarrow 0} MME(p) = \infty$ with $\frac{1+\beta-\alpha_0}{\beta} \in [1 - \frac{\alpha_0-1}{\beta}, \frac{1}{\alpha_0}] \subseteq (0, 1]$ and for $1 + \beta < \alpha_0$ we have $\lim_{p \rightarrow 0} MME(p) = 0$.

Remark 3.3 A few consequences of Corollary 3.2 are illustrated below along with some examples.

- (a) When $1 + \beta > \alpha_0$, although the quantity $MME(p)$ increases as $p \downarrow 0$, the rate of increase is slower than a linear function.
- (b) Let $\mathbf{Z} \in \mathcal{MRV}(\alpha, b, \nu, \mathbb{E})$. Suppose Z_1 and Z_2 are independent and $\bar{F}_{Z_1} \in \mathcal{RV}_{-\alpha}$ then by Karamata's Theorem,

$$MME(p) \sim \frac{1}{\alpha - 1} \text{VaR}_{1-p}(Z_2) P(Z_1 > \text{VaR}_{1-p}(Z_2)) \quad (p \downarrow 0).$$

This is a special case of Theorem 3.1.

- (c) In financial risk management, no doubt the most famous copula model is the *Gaussian copula*:

$$C_{\Phi, \rho}(u, v) = \Phi_2(\Phi^\leftarrow(u), \Phi^\leftarrow(v)) \quad \text{for } (u, v) \in [0, 1]^2,$$

where Φ is the standard-normal distribution function and Φ_2 is a bivariate normal distribution function with standard normally distributed margins and correlation ρ . Then the survival copula satisfies:

$$\hat{C}_{\Phi, \rho}(u, u) = C_{\Phi, \rho}(u, u) \sim u^{\frac{2}{\rho+1}} \ell(u) \quad (u \rightarrow 0)$$

(cf. Reiss (1989); Ledford and Tawn (1997)). Suppose (Z_1, Z_2) has identical Pareto marginal distributions with common parameter $\alpha > 0$ and a dependence structure given by a Gaussian copula $C_{\Phi, \rho}(u, v)$

with $\rho \in (-1, 1)$. Now we can check that $(Z_1, Z_2) \in \mathcal{MRV}(\alpha, b, \nu, \mathbb{E})$ with asymptotic tail independence and $(Z_1, Z_2) \in \mathcal{MRV}(\alpha_0, b_0, \nu_0, \mathbb{E}_0)$ with

$$\alpha_0 = \frac{2\alpha}{1+\rho} \quad \text{and} \quad \nu_0((x, \infty) \times (y, \infty)) = x^{-\frac{\alpha}{1+\rho}} y^{-\frac{\alpha}{1+\rho}}, \quad x, y > 0.$$

Hence, for $\rho \in (1 - \frac{2}{\alpha+1}, 1)$ we have $\lim_{p \rightarrow 0} \text{MME}(p) = \infty$. In this model, we Assumptions A and (B1) are satisfied when $\alpha > 1 + \rho$ and $\alpha > 1$. We can also check that (B2) is not satisfied. Consequently, we can find estimates for MME but not for MES in this example.

- (d) Suppose (Z_1, Z_2) has identical Pareto marginal distributions with parameter $\alpha > 0$ and a dependence structure given by a *Marshall-Olkin survival copula*:

$$\hat{C}_{\gamma_1, \gamma_2}(u, v) = uv \min(u^{-\gamma_1}, v^{-\gamma_2}) \quad \text{for } (u, v) \in [0, 1]^2,$$

for some $\gamma_1, \gamma_2 \in (0, 1)$. We can check that in this model, we have $(Z_1, Z_2) \in \mathcal{MRV}(\alpha, b, \nu, \mathbb{E})$ with asymptotic tail independence and $(Z_1, Z_2) \in \mathcal{MRV}(\alpha_0, b_0, \nu_0, \mathbb{E}_0)$ with

$$\alpha_0 = \alpha \max(2 - \gamma_1, 2 - \gamma_2) \quad \text{and} \quad \nu_0((x, \infty) \times (y, \infty)) = \begin{cases} x^{-\alpha(1-\gamma_1)} y^{-\alpha}, & \gamma_1 < \gamma_2, \\ x^{-\alpha} y^{-\alpha} \max(x, y)^{-\alpha\gamma_1}, & \gamma_1 = \gamma_2, \\ x^{-\alpha} y^{-\alpha(1-\gamma_2)}, & \gamma_1 > \gamma_2, \end{cases} \quad x, y > 0.$$

Then $\min(\gamma_1, \gamma_2) \in (1 - 1/\alpha, 1)$ implies $\lim_{p \rightarrow 0} \text{MME}(p) = \infty$. Moreover this model satisfies Assumptions A and (B1) when $\gamma_1 \geq \gamma_2$. Unfortunately again, (B2) is not satisfied.

Example 3.4 In this example we illustrate the influence of the tail behavior of the marginals as well as the dependence structure on the asymptotic behavior of the MME. Assume that $\mathbf{Z} = (Z_1, Z_2) \in [0, \infty)^2$ satisfies Assumptions (A1)-(A4). We compare the following tail independent and tail dependent models:

- (D) Tail dependent model: Additionally \mathbf{Z} is tail dependent implying $R \neq 0$ and satisfies (1.5). We denote its Marginal Mean Excess by MME^D .
- (ID) Tail independent model: Additionally \mathbf{Z} is *asymptotically* tail independent satisfying (A5), (B1) and $1 + \beta > \alpha_0$. Its Marginal Mean Excess we denote by MME^I .
- (a) Suppose Z_1, Z_2 are identically distributed. Since $t/b_0^{\leftarrow}(b_2(t)) \in \mathcal{RV}_{1-\alpha_0/\alpha}$ and $1 - \alpha_0/\alpha \leq 0$ we get

$$\lim_{p \rightarrow 0} \frac{\text{MME}^I(p)}{\text{MME}^D(p)} = \lim_{p \rightarrow 0} \frac{1}{pb_0^{\leftarrow}(b_2(1/p))} = 0.$$

This means in the asymptotically tail independent case the Marginal Mean Excess increases slower to infinity than in the asymptotically tail dependent case as expected.

- (b) Suppose Z_1, Z_2 are not identically distributed and for some finite constant $C > 0$

$$P(Z_2 > t) \sim CP(Z_1 > t, Z_2 > t) \quad (t \rightarrow \infty),$$

i.e. not only $\mathbf{Z} \in \mathcal{MRV}(\alpha_0, b_0, \nu_0, \mathbb{E}_0)$ but also $Z_2 \in \mathcal{RV}_{-\alpha_0}$ and Z_1 is heavier tailed than Z_2 . Then

$$\lim_{t \rightarrow \infty} \frac{b_0^{\leftarrow}(b_2(t))}{t} = \lim_{t \rightarrow \infty} \frac{1}{tP(Z_1 > b_2(t), Z_2 > b_2(t))} = \lim_{t \rightarrow \infty} \frac{C}{tP(Z_2 > b_2(t))} = C.$$

Thus,

$$\lim_{p \rightarrow 0} \frac{1}{\text{VaR}_{1-p}(Z_2)} \text{MME}^I(p) = \lim_{p \rightarrow 0} \frac{\text{MME}^I(p)}{b_2(1/p)} = C \int_0^\infty \nu_0((x, \infty) \times (1, \infty)) dx$$

and $\text{MME}^I(\cdot)$ is regularly varying of index $-\frac{1}{\alpha_0}$ at 0. In this example Z_2 is lighter tailed than Z_1 , and hence, once again we find that in the asymptotically tail independent case the Marginal Mean Excess MME^I increases at a slower rate to infinity than the Marginal Mean Excess MME^D in the asymptotically tail dependent case.

3.2. Asymptotic behavior of the MES

Here we derive analogous results for the Marginal Expected Shortfall.

Theorem 3.5. *Suppose $\mathbf{Z} = (Z_1, Z_2)$ satisfies Assumptions A and B. Then*

$$\lim_{p \downarrow 0} \frac{pb_0^{\leftarrow}(\text{VaR}_{1-p}(Z_2))}{\text{VaR}_{1-p}(Z_2)} \text{MES}(p) = \lim_{p \downarrow 0} \frac{pb_0^{\leftarrow}(b_2(1/p))}{b_2(1/p)} \text{MES}(p) = \int_0^\infty \nu_0((x, \infty) \times (1, \infty)) \, dx. \quad (3.2)$$

Moreover, $0 < \int_0^\infty \nu_0((x, \infty) \times (1, \infty)) \, dx < \infty$.

The proof of Theorem 3.5 requires further condition (B2) which we can be avoided in Theorem 3.1.

Proof. The proof is similar to that of Theorem 3.1 which we discussed in detail. As in Theorem 3.1 we rewrite

$$\frac{pb_0^{\leftarrow}(b_2(1/p))}{b_2(1/p)} \text{MES}(p) = \frac{\bar{F}_{Z_2}(t)b_0^{\leftarrow}(t)}{t} \mathbb{E}(Z_1|Z_2 > t) = \left[\int_0^{1/M} + \int_{1/M}^M + \int_M^\infty \right] \frac{P(Z_1 > tx, Z_2 > t)}{P(Z_1 > t, Z_2 > t)} \, dx.$$

We can then conclude the statement from (B2) and similar arguments as in the proof of Theorem 3.1. \square

A similar comparison can be made between the asymptotic behavior of the Marginal Expected Shortfall for the tail independent and tail dependent case as we have done in Example 3.4 for the Marginal Mean Excess.

Remark 3.6 Define

$$a(t) := \frac{\frac{1}{t}b_0^{\leftarrow}(b_2(t))}{b_2(t)}.$$

Then $\lim_{t \rightarrow \infty} a(t) = 0$ is equivalent to $\lim_{t \rightarrow \infty} \frac{P(Z_2 > t)}{tP(Z_1 > t, Z_2 > t)} = 0$. Hence, a consequence of (2.3) and (B2) is that $\lim_{t \rightarrow \infty} a(t) = 0$ and finally, $\lim_{p \downarrow 0} \text{MES}(p) = \infty$. Again a sufficient assumption for $\lim_{t \rightarrow \infty} a(t) = 0$ is $\bar{F}_{Z_2} \in \mathcal{RV}_{-\beta}$ with $\alpha_0 < \beta + 1$ and a necessary condition is $\alpha_0 \leq \beta + 1$ (cf. Lemma 2.10).

Remark 3.7 In this study we have only considered a non-negative random variable Z_1 while computing $\text{MES}(p) = \mathbb{E}(Z_1|Z_2 > \text{VaR}_{1-p}(Z_2))$. In the case we have a real-valued random variable Z_1 , we can represent it as $Z_1 = Z_1^+ - Z_1^-$ where $Z_1^+ = \max(Z_1, 0)$ and $Z_1^- = \max(-Z_1, 0)$. Here both Z_1^+ and Z_1^- are non-negative and hence can be dealt with separately. The limit results will depend on the separate dependence structure and tail behaviors of (Z_1^+, Z_2) and (Z_1^-, Z_2) .

4. Estimation of MME and MES

4.1. Empirical estimators for the MME and the MES

4.1.1. Empirical estimator for the MME

We begin by looking at the behavior of the empirical estimator

$$\widehat{\text{MME}}_{\text{emp},n}(k/n) := \frac{1}{k} \sum_{i=1}^n (Z_i^{(1)} - Z_{(k:n)}^{(2)})_+ \mathbb{1}_{\{Z_i^{(2)} > Z_{(k:n)}^{(2)}\}}$$

of the quantity $\text{MME}(k/n) = \mathbb{E}((Z_1 - b_2(n/k))_+ | Z_2 > b_2(n/k))$ with $k < n$. The following theorem shows that the empirical estimator is consistent in probability.

Proposition 4.1. *Let the assumptions of Theorem 3.1 hold, and let $\bar{F}_{Z_2} \in \mathcal{RV}_{-\beta}$ for some $\alpha \leq \beta \leq \alpha_0$. Furthermore, let $k = k(n)$ be a sequence of integers satisfying $k \rightarrow \infty$, $k/n \rightarrow 0$ and $b_0^{\leftarrow}(b_2(n/k))/n \rightarrow 0$ as $n \rightarrow \infty$ (note that this is trivially satisfied if $b_0 = b_2$). We denote by $Z_{(1:n)}^{(2)} \geq \dots \geq Z_{(n:n)}^{(2)}$ the order statistic of the sample $Z_1^{(2)}, \dots, Z_n^{(2)}$ in decreasing order.*

(a) *Then, as $n \rightarrow \infty$,*

$$\frac{b_0^{\leftarrow}(b_2(n/k))}{b_2(n/k)} \frac{1}{n} \sum_{i=1}^n (Z_i^{(1)} - Z_{(k:n)}^{(2)})_+ \mathbb{1}_{\{Z_i^{(2)} > Z_{(k:n)}^{(2)}\}} \xrightarrow{P} \int_1^\infty \nu_0((x, \infty) \times (1, \infty)) dx.$$

(b) *In particular, we have $\frac{\widehat{MME}_{\text{emp},n}(k/n)}{MME(k/n)} \xrightarrow{P} 1$ as $n \rightarrow \infty$.*

To prove this theorem we use the following lemma.

Lemma 4.2. *Let the assumptions of Proposition 4.1 hold. Define for $y > 0$,*

$$\begin{aligned} E_n(y) &:= \frac{b_0^{\leftarrow}(b_2(n/k))}{b_2(n/k)} \frac{1}{n} \sum_{i=1}^n (Z_i^{(1)} - b_2(n/k)y)_+ \mathbb{1}_{\{Z_i^{(2)} > b_2(n/k)y\}}, \\ E(y) &:= \int_y^\infty \nu_0((x, \infty) \times (y, \infty)) dx. \end{aligned}$$

Then $E(y) = y^{1-\alpha_0} E(1)$ and as $n \rightarrow \infty$,

$$(E_n(y))_{y \geq 1/2} \xrightarrow{P} (E(y))_{y \geq 1/2} \quad \text{in } \mathbb{D}([1/2, \infty), (0, \infty)),$$

where by $\mathbb{D}(I, \mathbb{E}^)$ we denote the space of càdlàg functions from $I \rightarrow \mathbb{E}^*$.*

Proof.

(a) We already know from (Resnick, 2007, Theorem 5.3(ii)), $\mathbf{Z} \in \mathcal{MRV}(\alpha_0, b_0, \nu_0, \mathbb{E}_0)$ and $b_0^{\leftarrow}(b_2(n/k))/n \rightarrow 0$ that as $n \rightarrow \infty$,

$$\nu_0^{(n)} := \frac{b_0^{\leftarrow}(b_2(n/k))}{n} \sum_{i=1}^n \epsilon_{\left(\frac{Z_i^{(1)}}{b_2(n/k)}, \frac{Z_i^{(2)}}{b_2(n/k)}\right)} \Rightarrow \nu_0 \quad \text{in } \mathbb{M}_+(\mathbb{E}_0). \quad (4.1)$$

Note that

$$E_n(y) = \int_y^\infty \nu_0^{(n)}((x, \infty) \times (y, \infty)) dx = \frac{b_0^{\leftarrow}(b_2(n/k))}{b_2(n/k)} \frac{1}{n} \sum_{i=1}^n (Z_i^{(1)} - b_2(n/k)y)_+ \mathbb{1}_{\{Z_i^{(2)} > b_2(n/k)y\}}.$$

Hence, the statement of the lemma is equivalent to

$$\left(\int_y^\infty \nu_0^{(n)}((x, \infty) \times (y, \infty)) dx \right)_{y \geq \frac{1}{2}} \xrightarrow{P} (E(y))_{y \geq 1/2} \quad \text{in } \mathbb{D}([1/2, \infty), (0, \infty)). \quad (4.2)$$

We will prove (4.2) by a convergence-together argument.

Step 1. First we prove that $E(y) = y^{1-\alpha_0} E(1)$. From Theorem 3.1 we have,

$$\frac{b_0^{\leftarrow}(b_2(n/k))}{b_2(n/k)} \mathbb{E}((Z_1 - b_2(n/k)y)_+ \mathbb{1}_{\{Z_2 > b_2(n/k)y\}})$$

$$\begin{aligned}
&= \int_1^\infty \frac{P(Z_1 > xb_2(n/k), Z_2 > b_2(n/k)y)}{P(Z_1 > b_2(n/k), Z_2 > b_2(n/k))} dx \\
&= y \frac{P(Z_1 > b_2(n/k)y, Z_2 > b_2(n/k)y)}{P(Z_1 > b_2(n/k), Z_2 > b_2(n/k))} \int_1^\infty \frac{P(Z_1 > x(b_2(n/k)y), Z_2 > b_2(n/k)y)}{P(Z_1 > b_2(n/k)y, Z_2 > b_2(n/k)y)} dx \\
&\xrightarrow{n \rightarrow \infty} y^{1-\alpha_0} \int_1^\infty \nu_0((x, \infty) \times (1, \infty)) dx = y^{1-\alpha_0} E(1).
\end{aligned} \tag{4.3}$$

On the other hand, in a similar manner as in Theorem 3.1, we can exchange the integral and the limit such that using (2.1) we obtain

$$\begin{aligned}
\frac{b_0^-(b_2(n/k))}{b_2(n/k)} \mathbb{E}((Z_1 - b_2(n/k)y)_+ \mathbb{1}_{\{Z_2 > b_2(n/k)y\}}) &= \int_1^\infty \frac{P(Z_1 > xb_2(n/k), Z_2 > b_2(n/k)y)}{P(Z_1 > b_2(n/k), Z_2 > b_2(n/k))} dx \\
&\xrightarrow{n \rightarrow \infty} \int_1^\infty \nu_0((x, \infty) \times (y, \infty)) dx = E(y).
\end{aligned} \tag{4.4}$$

Since (4.3) and (4.4) must be equal, we have $E(y) = y^{1-\alpha_0} E(1)$.

Step 2. In the first step we prove that for any $y \geq 1/2$ and $M > 0$, as $n \rightarrow \infty$,

$$E_n^{(M)}(y) := \int_1^M \nu_0^{(n)}((x, \infty) \times (y, \infty)) dx \xrightarrow{P} \int_1^M \nu_0((x, \infty) \times (y, \infty)) dx =: E^{(M)}(y). \tag{4.5}$$

Define the function $f_{M,y} : \mathbb{E}_0 \rightarrow [0, M]$ as $f_{M,y}(z_1, z_2) = (\min(z_1, M) - y) \mathbb{1}_{\{z_1 > y, z_2 > y\}}$ which is continuous, bounded and has compact support on \mathbb{E}_0 and for any $y \geq 1/2$ define $F_{M,y} : \mathbb{M}_+(\mathbb{E}_0) \rightarrow \mathbb{R}_+$ as

$$m \mapsto \int_{\mathbb{E}_0} f_{M,y}(z_1, z_2) m(dz_1, dz_2).$$

Here m is a continuous map on $\mathbb{M}_+(\mathbb{E}_0)$ under the vague topology. Hence, using a continuous mapping theorem and (4.1) we get, as $n \rightarrow \infty$,

$$\int_1^M \nu_0^{(n)}((x, \infty) \times (y, \infty)) dx = F_{M,y}(\nu_0^{(n)}) \Rightarrow F_{M,y}(\nu_0) = \int_1^M \nu_0((x, \infty) \times (y, \infty)) dx \tag{4.6}$$

in \mathbb{R}_+ . Since the right hand side is deterministic, the convergence holds in probability as well.

Step 3. Using B,

$$\begin{aligned}
\mathbb{E} \left(\sup_{y \geq \frac{1}{2}} \int_M^\infty \nu_0^{(n)}((x, \infty) \times (y, \infty)) dx \right) &\leq \mathbb{E} \left(\int_M^\infty \nu_0^{(n)}((x, \infty) \times (1/2, \infty)) dx \right) \\
&= b_0^-(b_2(n/k)) \int_M^\infty P(Z_1 > xb_2(n/k), Z_2 > b_2(n/k)/2) dx \\
&= \int_M^\infty \frac{P(Z_1 > xb_2(n/k), Z_2 > b_2(n/k)/2)}{P(Z_1 > b_2(n/k), Z_2 > b_2(n/k))} dx \xrightarrow{n \rightarrow \infty, M \rightarrow \infty} 0.
\end{aligned}$$

Step 4. Hence, a convergence-together argument (see (Resnick, 2007, Theorem 3.5)), Step 2, Step 3 and $E^{(M)}(y) \rightarrow E(y)$ as $M \rightarrow \infty$ result in $E_n(y) \xrightarrow{P} E(y)$ as $n \rightarrow \infty$.

Step 5. From Step 1, the function $E : [1/2, \infty) \rightarrow (0, E(1/2)]$ is a decreasing, continuous function as well as a bijection. Let E^{-1} denote its inverse and define for $m \in \mathbb{N}$ and $k = 1, \dots, m$,

$$y_{m,k} := E^{-1} \left(E(1/2) \frac{k}{m} \right).$$

As in the proof of the Glivenko-Cantelli-Theorem (cf. (Billingsley, 1995, Theorem 20.6)) we have

$$\sup_{y \geq 1/2} |E_n(y) - E(y)| \leq \frac{E(1/2)}{m} + \sup_{k=1, \dots, m} |E_n(y_{m,k}) - E(y_{m,k})|.$$

Let $\epsilon > 0$. Choose $m \in \mathbb{N}$ such that $m > 2E(1/2)/\epsilon$. Then

$$\begin{aligned} P\left(\sup_{y \geq 1/2} |E_n(y) - E(y)| > \epsilon\right) &\leq P\left(\sup_{k=1, \dots, m} |E_n(y_{m,k}) - E(y_{m,k})| > E(1/2)m^{-1}\right) \\ &\leq \sum_{k=1}^m P(|E_n(y_{m,k}) - E(y_{m,k})| > E(1/2)m^{-1}) \xrightarrow{n \rightarrow \infty} 0, \end{aligned}$$

where we used $E_n(y_{m,k}) \xrightarrow{P} E(y_{m,k})$ as $n \rightarrow \infty$ for any $k = 1, \dots, m$, $m \in \mathbb{N}$ by Step 4. Hence, we can conclude the statement. \square

Proof of Proposition 4.1.

(a) By assumption, $\bar{F}_{Z_2} \in \mathcal{RV}_{-\beta}$. From (Resnick, 2007, p. 82) we know that

$$\left(\frac{Z_{(\lceil ky \rceil : n)}^{(2)}}{b_2(n/k)}\right)_{y>0} \xrightarrow{P} (y^{-\frac{1}{\beta}})_{y>0} \quad \text{in } \mathbb{D}((0, \infty], (0, \infty))$$

and in particular this and Lemma 4.2 results in

$$\left((E_n(y))_{y \geq \frac{1}{2}}, \left(\frac{Z_{(\lceil ky \rceil : n)}^{(2)}}{b_2(n/k)}\right)_{y>0}\right) \xrightarrow{P} ((E(y))_{y \geq \frac{1}{2}}, (y^{-\frac{1}{\beta}})_{y>0}) \quad \text{in } \mathbb{D}([1/2, \infty), (0, \infty)) \times \mathbb{D}((0, \infty], (0, \infty)).$$

Let $\mathbb{D}^\downarrow((0, 2^\beta], [1/2, \infty))$ be a subfamily of $\mathbb{D}((0, 2^\beta], [1/2, \infty))$ consisting of non-increasing functions. Let us similarly define $\mathbb{C}^\downarrow((0, 2^\beta], [1/2, \infty))$. Define the map $\varphi : \mathbb{D}([1/2, \infty), (0, \infty)) \times \mathbb{D}^\downarrow((0, 2^\beta], [1/2, \infty))$ with $(f, g) \mapsto f \circ g$. From (Whitt, 2002, Theorem 13.2.2), we already know that φ restricted to $\mathbb{D}([1/2, \infty), (0, \infty)) \times \mathbb{C}^\downarrow((0, 2^\beta], [1/2, \infty))$ is continuous. Thus, we can apply a continuous mapping theorem and obtain as $n \rightarrow \infty$,

$$\left(E_n \left(\frac{Z_{(\lceil ky \rceil : n)}^{(2)}}{b_2(n/k)}\right)\right)_{y \in (0, 2^\beta]} \xrightarrow{P} (E(y^{-\frac{1}{\beta}}))_{y \in (0, 2^\beta]} \quad \text{in } \mathbb{D}((0, 2^\beta], (0, \infty)).$$

As a special case we get the marginal convergence as $n \rightarrow \infty$,

$$\frac{b_0^-(b_2(n/k))}{b_2(n/k)} \frac{1}{n} \sum_{i=1}^n Z_i^{(1)} \mathbb{1}_{\{Z_i^{(2)} > Z_{(k:n)}^{(2)}\}} = E_n \left(\frac{Z_{(k:n)}^{(2)}}{b_2(n/k)}\right) \xrightarrow{P} E(1) = \int_0^\infty \nu_0((x, \infty) \times (1, \infty)) dx.$$

(b) Finally, from part (a) and Theorem 3.1 we have

$$\frac{\frac{1}{k} \sum_{i=1}^n Z_i^{(1)} \mathbb{1}_{\{Z_i^{(2)} > Z_{(k:n)}^{(2)}\}}}{\text{MME}(k/n)} = \frac{\frac{b_0^-(b_2(n/k))}{b_2(n/k)} \frac{1}{n} \sum_{i=1}^n Z_i^{(1)} \mathbb{1}_{\{Z_i^{(2)} > Z_{(k:n)}^{(2)}\}}}{\frac{\frac{k}{n} b_0^-(b_2(n/k))}{b_2(n/k)} \text{MME}(k/n)} \xrightarrow{P} \frac{\int_0^\infty \nu_0((x, \infty) \times (1, \infty)) dx}{\int_0^\infty \nu_0((x, \infty) \times (1, \infty)) dx} = 1,$$

which is what we needed to show. \square

4.1.2. Empirical estimator for the MES

An analogous result holds for the empirical estimator

$$\widehat{\text{MES}}_{\text{emp}, n}(k/n) := \frac{1}{k} \sum_{i=1}^n Z_i^{(1)} \mathbb{1}_{\{Z_i^{(2)} > Z_{(k:n)}^{(2)}\}}$$

of $\text{MES}(k/n) = \mathbb{E}(Z_1 | Z_2 > b_2(n/k))$ where $k < n$.

Proposition 4.3. *Let the assumptions of Theorem 3.5 hold, and let $\bar{F}_{Z_2} \in \mathcal{RV}_{-\beta}$ for some $\alpha \leq \beta \leq \alpha_0$. Furthermore, let $k = k(n)$ be a sequence of integers satisfying $k \rightarrow \infty$, $k/n \rightarrow 0$ and $b_0^{\leftarrow}(b_2(n/k))/n \rightarrow 0$ as $n \rightarrow \infty$.*

(a) *Then, as $n \rightarrow \infty$,*

$$\frac{b_0^{\leftarrow}(b_2(n/k))}{b_2(n/k)} \frac{1}{n} \sum_{i=1}^n Z_i^{(1)} \mathbb{1}_{\{Z_i^{(2)} > Z_{(k:n)}^{(2)}\}} \xrightarrow{P} \int_0^\infty \nu_0((x, \infty) \times (1, \infty)) dx.$$

(b) *In particular, $\frac{\widehat{MES}_{emp,n}(k/n)}{MES(k/n)} \xrightarrow{P} 1$, as $n \rightarrow \infty$*

The proof of the theorem is analogous to that of the proof of Proposition 4.1 based on the following version of Lemma 4.2. Hence, we skip the details.

Lemma 4.4. *Let the assumptions of Proposition 4.3 hold. Define for $y > 0$,*

$$\begin{aligned} E_n(y) &:= \frac{b_0^{\leftarrow}(b_2(n/k))}{b_2(n/k)} \frac{1}{n} \sum_{i=1}^n Z_i^{(1)} \mathbb{1}_{\{Z_i^{(2)} > b_2(n/k)y\}}, \\ E(y) &:= \int_0^\infty \nu_0((x, \infty) \times (y, \infty)) dx. \end{aligned}$$

Then $E(y) = y^{1-\alpha_0} E(1)$ and as $n \rightarrow \infty$,

$$(E_n(y))_{y \geq 1/2} \xrightarrow{P} (E(y))_{y \geq 1/2} \quad \text{in } \mathbb{D}([1/2, \infty), (0, \infty)).$$

Proof. The only differences between the proofs of Lemma 4.2 and Lemma 4.4 are that in the proof of Lemma 4.4 we use $E_n^{(M)}(y) := \int_{\frac{1}{M}}^M \nu_0^{(n)}((x, \infty) \times (y, \infty)) dx$ and that in Step 3, we have

$$\lim_{M \rightarrow \infty} \lim_{n \rightarrow \infty} \mathbb{E} \left(\sup_{y \geq \frac{1}{2}} \left[\int_0^{\frac{1}{M}} + \int_M^\infty \right] \nu_0^{(n)}((x, \infty) \times (y, \infty)) dx \right) = 0$$

where Assumption (B2) has to be used. □

4.2. Estimators for the MME and the MES based on extreme value theory

In certain situations we might be interested in estimating $\text{MME}(p)$ or $\text{MES}(p)$ in a region where no data are available. Since empirical estimators would not work in such a case we can resort to extrapolation via extreme value theory. We start with a motivation for the definition of the estimator before we provide its asymptotic properties. For the rest of this section we make the following assumption.

Assumption D $\bar{F}_{Z_2} \in \mathcal{RV}_{-\beta}$ for $\alpha \leq \beta \leq \alpha_0 < \beta + 1$.

D guarantees that $\lim_{t \rightarrow \infty} a(t) = 0$ (cf. Remark 3.6). The extrapolation idea is that for all $p \geq k/n$, we estimate $\text{MME}(p)$ empirically since sufficient data are available in this region; on the other hand for $p < k/n$ we will use an extrapolating extreme-value technique. For notational convenience, define the function

$$a(t) := \frac{1}{t} b_0^{\leftarrow}(b_2(t))/b_2(t).$$

Since $b_0^{\leftarrow} \in \mathcal{RV}_{\alpha_0}$ and $b_2 \in \mathcal{RV}_{1/\beta}$, we have $a \in \mathcal{RV}_{\frac{\alpha_0 - \beta - 1}{\beta}}$. Now, let $k := k(n)$ be a sequence of integers so that $k/n \rightarrow 0$ as $n \rightarrow \infty$. From Theorem 3.1 we already know that

$$\lim_{p \downarrow 0} a(1/p) \text{MME}(p) = \int_1^\infty \nu_0((x, \infty) \times (1, \infty)) dx = \lim_{n \rightarrow \infty} a(n/k) \text{MME}(k/n).$$

Hence,

$$\text{MME}(p) \sim \frac{a(n/k)}{a(1/p)} \text{MME}(k/n) \sim \left(\frac{k}{np}\right)^{\frac{\beta - \alpha_0 + 1}{\beta}} \text{MME}(k/n) \quad (p \downarrow 0). \quad (4.7)$$

If we plug in the estimators $\hat{\alpha}_{0,n}$, $\hat{\beta}_n$ and $\widehat{\text{MME}}_n(k/n)$ for α_0 , β and $\text{MME}(k/n)$ respectively in (4.7) we obtain an estimator for $\text{MME}(p)$ given by

$$\widehat{\text{MME}}_n(p) = \left(\frac{k}{np}\right)^{\frac{\hat{\beta}_n - \hat{\alpha}_{0,n} + 1}{\hat{\beta}_n}} \widehat{\text{MME}}_{\text{emp},n}(k/n).$$

Similarly, we may obtain an estimator of $\text{MES}(p)$ given by

$$\widehat{\text{MES}}_n(p) = \left(\frac{k}{np}\right)^{\frac{\hat{\beta}_n - \hat{\alpha}_{0,n} + 1}{\hat{\beta}_n}} \widehat{\text{MES}}_{\text{emp},n}(k/n).$$

If $\beta > \alpha$ then the parameter α , the index of regular variation of Z_1 , is surprisingly not necessary for the estimation of either Marginal Mean Excess or Marginal Expected Shortfall.

Theorem 4.5. *Let Assumptions A, (B1) and D hold. Furthermore, let $k = k(n)$ be a sequence of integers satisfying $k \rightarrow \infty$, $k/n \rightarrow 0$ as $n \rightarrow \infty$. Moreover, $p_n \in (0, 1)$ is a sequence of constants with $p_n \downarrow 0$ and $np_n = o(k)$ as $n \rightarrow \infty$. Let $\hat{\alpha}_{0,n}$ and $\hat{\beta}_n$ be estimators for α_0 and β , respectively such that*

$$\log \frac{k}{np_n} (\hat{\alpha}_{0,n} - \alpha_0) \xrightarrow{P} 0 \quad \text{and} \quad \log \frac{k}{np_n} (\hat{\beta}_n - \beta) \xrightarrow{P} 0 \quad (n \rightarrow \infty). \quad (4.8)$$

(a) Then $\frac{\widehat{\text{MME}}_n(p_n)}{\text{MME}(p_n)} \xrightarrow{P} 1$ as $n \rightarrow \infty$.

(b) Additionally, if Assumption (B2) is satisfied then $\frac{\widehat{\text{MES}}_n(p_n)}{\text{MES}(p_n)} \xrightarrow{P} 1$ as $n \rightarrow \infty$.

Proof. (a) Rewrite

$$\begin{aligned} \frac{\widehat{\text{MME}}_n(p_n)}{\text{MME}(p_n)} &= \frac{\left(\frac{k}{np_n}\right)^{\frac{\hat{\beta}_n - \hat{\alpha}_{0,n} + 1}{\hat{\beta}_n}} \widehat{\text{MME}}_{\text{emp},n}(k/n)}{\text{MME}(p_n)} \\ &= \frac{\widehat{\text{MME}}_{\text{emp},n}(k/n)}{\text{MME}(k/n)} \frac{a(n/k) \text{MME}(k/n)}{a(1/p_n) \text{MME}(p_n)} \frac{\frac{a(1/p_n)}{a(n/k)} \left(\frac{k}{np_n}\right)^{\frac{\hat{\beta}_n - \hat{\alpha}_{0,n} + 1}{\hat{\beta}_n}}}{\left(\frac{np_n}{k}\right)^{\frac{\beta - \alpha_0 + 1}{\beta}} \left(\frac{k}{np_n}\right)^{\frac{\beta - \alpha_0 + 1}{\beta}}} \\ &=: I_1(n) \cdot I_2(n) \cdot I_3(n) \cdot I_4(n). \end{aligned}$$

An application of Proposition 4.1 implies $I_1(n) = \frac{\widehat{\text{MME}}_{\text{emp},n}(k/n)}{\text{MME}(k/n)} \xrightarrow{P} 1$ as $n \rightarrow \infty$. For the second term $I_2(n)$, using Theorem 3.1 we have

$$I_2(n) = \frac{\frac{\frac{k}{n} b_0^{\leftarrow}(b_2(n/k))}{b_2(n/k)} \text{MME}(k/n)}{\frac{p_n b_0^{\leftarrow}(b_2(1/p_n))}{b_2(1/p_n)} \text{MME}(p_n)} \xrightarrow{P} 1.$$

Since $a \in \mathcal{RV}_{\frac{\alpha_0 - \beta - 1}{\beta}}$, $k/n \rightarrow 0$ and $p_n < k/n$ we obtain $\lim_{n \rightarrow \infty} I_3(n) = 1$ as well. For the last term $I_4(n)$ we use the representation

$$\frac{\left(\frac{k}{np_n}\right)^{\frac{\hat{\beta}_n - \hat{\alpha}_{0,n} + 1}{\hat{\beta}_n}}}{\left(\frac{k}{np_n}\right)^{\frac{\beta - \alpha_0 + 1}{\beta}}} = \exp\left(\left(\frac{1 - \hat{\alpha}_{0,n}}{\hat{\beta}_n} - \frac{1 - \alpha_0}{\beta}\right) \log\left(\frac{k}{np_n}\right)\right),$$

and

$$\frac{1 - \hat{\alpha}_{0,n}}{\hat{\beta}_n} - \frac{1 - \alpha_0}{\beta} = (\beta - \hat{\beta}_n) \frac{1 - \hat{\alpha}_{0,n}}{\hat{\beta}_n \beta} + (\alpha_0 - \hat{\alpha}_{0,n}) \frac{1}{\beta}.$$

Since by assumption (4.8) we have $\hat{\alpha}_{0,n} \xrightarrow{P} \alpha$, $\hat{\beta}_n \xrightarrow{P} \beta$, using a continuous mapping theorem we get,

$$\log \frac{k}{np_n} (\beta - \hat{\beta}_n) \frac{1 - \hat{\alpha}_{0,n}}{\hat{\beta}_n \beta} + \log \frac{k}{np_n} (\alpha_0 - \hat{\alpha}_{0,n}) \frac{1}{\beta} \xrightarrow{P} 0.$$

Hence, we conclude that $I_4(n) \xrightarrow{P} 1$ as $n \rightarrow \infty$ which completes the proof.

(b) This proof is analogous to (a) and hence is omitted here. \square

5. Simulation study

In this section, we study the developed estimators for different models. We simulate from models described in Section 2 and Section 3, estimate MME and MES values from the data and compare them with the actual values from the model. We also compare our estimator with a regular empirical estimator and observe that our estimator provides a smaller variance in most simulated examples. Moreover our estimator is scalable to smaller $p < 1/n$ where n is the sample size, which is infeasible for the empirical estimator.

5.1. Estimators and assumption checks

As an estimator of β , the index of regular variation of $Z^{(2)}$ we use the Hill-estimator based on the data $Z_1^{(2)}, \dots, Z_n^{(2)}$ whose order statistics is given by $Z_{(1:n)}^{(2)} \geq \dots \geq Z_{(n:n)}^{(2)}$. The estimator is

$$\hat{\beta}_n = \frac{1}{k_2} \sum_{i=1}^{k_2} [\log(Z_{(i:n)}^{(2)}) - \log(Z_{(k_1:n)}^{(2)})]$$

for some $k_2 := k_2(n) \in \{1, \dots, n\}$. Similarly, we use as estimator for α_0 , the index of hidden regular variation, the Hill-estimator based on the data $\min(Z_1^{(2)}, Z_1^{(2)}), \dots, \min(Z_n^{(1)}, Z_n^{(2)})$. Therefore, define $Z_i^{\min} = \min(Z_i^{(2)}, Z_i^{(2)})$ for $i \in \mathbb{N}$. The order statistic of $Z_1^{\min}, \dots, Z_n^{\min}$ is denoted as $Z_{(1:n)}^{\min} \geq \dots \geq Z_{(n:n)}^{\min}$. The Hill-estimator for α_0 is then

$$\hat{\alpha}_{0,n} = \frac{1}{k_0} \sum_{i=1}^{k_0} [\log(Z_{(i:n)}^{\min}) - \log(Z_{(k_2:n)}^{\min})]$$

for some $k_0 := k_0(n) \in \{1, \dots, n\}$.

Corollary 5.1. *Let Assumptions A and D hold. Furthermore, suppose the following conditions are satisfied:*

1. $\min(k, k_0, k_2) \rightarrow \infty$, $\max(k, k_0, k_2)/n \rightarrow 0$ as $n \rightarrow \infty$.

2. $p_n \in (0, 1)$ such that $p_n \downarrow 0$, $np_n = o(k)$ and $\log(k/(np_n)) = o(\min(\sqrt{k_0}, \sqrt{k_2}))$ as $n \rightarrow \infty$.
3. The second order conditions

$$\lim_{t \rightarrow \infty} \frac{\frac{b_0(tx)}{b_0(t)} - x^{1/\alpha_0}}{A_0(t)} = x^{1/\alpha_0} \frac{x^{\rho_0} - 1}{\rho_0} \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{\frac{b_2(tx)}{b_2(t)} - x^{1/\beta}}{A_2(t)} = x^{1/\beta} \frac{x^{\rho_2} - 1}{\rho_2}, \quad x > 0,$$

where $\rho_0, \rho_2 \leq 0$ are constants and A_0, A_2 are positive or negative functions hold.

4. $\lim_{t \rightarrow \infty} A_0(t) = \lim_{t \rightarrow \infty} A_2(t) = 0$.
5. There exist finite constants λ_0, λ_2 such that

$$\lim_{n \rightarrow \infty} \sqrt{k_0} A_0\left(\frac{n}{k_0}\right) = \lambda_0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \sqrt{k_2} A_2\left(\frac{n}{k_2}\right) = \lambda_2.$$

Then (4.8) is satisfied.

Proof. From (de Haan and Ferreira, 2006, Theorem 3.2.5) we know that $\sqrt{k_2}(\beta - \hat{\beta}_n) \xrightarrow{\mathcal{D}} \mathcal{N}$ as $n \rightarrow \infty$ where \mathcal{N} is a normally distributed random variable. In particular, $\hat{\beta}_n \xrightarrow{P} \beta$ as $n \rightarrow \infty$. The analogous result holds for $\hat{\alpha}_{0,n}$ as well. Since by assumption $\log(k/np_n) = o(\sqrt{k_i})$ ($i = 0, 2$), we obtain

$$\sqrt{k_0}(\hat{\alpha}_{0,n} - \alpha_0) \frac{\log \frac{k}{np_n}}{\sqrt{k_0}} \xrightarrow{P} 0 \quad \text{and} \quad \sqrt{k_2}(\hat{\beta}_n - \beta) \frac{\log \frac{k}{np_n}}{\sqrt{k_2}} \xrightarrow{P} 0, \quad (n \rightarrow \infty)$$

which is condition (4.8). □

In our simulation study we take $k = k_1 = k_2$.

Remark 5.2 An alternative to the Hill estimator is the probability weighted moment estimator based on the block maxima method which is under some regularity condition consistent and asymptotically normally distributed as presented in (Ferreira and de Haan, 2015, Theorem 2.3) and hence, satisfies (4.8). Moreover, the peaks-over-threshold (POT) method is a further option to estimate α_0, β which satisfies as well under some regularity conditions (4.8); for more details on the asymptotic behavior of estimators based on the POT method see Smith (1987).

5.2. Simulated Examples

First we use our methods on a few simulated examples.

Example 5.3 (GAUSSIAN COPULA) Suppose (Z_1, Z_2) has identical Pareto marginal distributions with common parameter $\alpha > 0$ and a dependence structure given by a Gaussian copula $C_{\Phi, \rho}(u, v)$ with $\rho \in (-1, 1)$ as given in Remark 3.3 (c). A further restriction from the same remark leads us to assume $\rho \in (1 - \frac{2}{\alpha+1}, 1)$ so that $\lim_{p \rightarrow 0} \text{MME}(p) = \infty$.

In the Gaussian copula model for specified p we can numerically compute the value of $\text{MME}(p)$. In our study we generate the above distribution for four sets of choices of parameters for each of which all the necessary conditions are satisfied:

- (a) $\alpha = 2, \rho = 0.9$. Hence $\alpha_0 = 2.1$.
- (b) $\alpha = 2, \rho = 0.5$. Hence $\alpha_0 = 2.67$.
- (c) $\alpha = 2.3, \rho = 0.8$. Hence $\alpha_0 = 2.55$.
- (d) $\alpha = 1.9, \rho = 0.8$. Hence $\alpha_0 = 2.11$.

The parameters α and α_0 are estimated using the Hill estimator which appears to estimate the parameters quite well; see Resnick (2007) for details. The estimated values $\hat{\alpha}$ and $\hat{\alpha}_0$ are used to compute estimated values of MME.

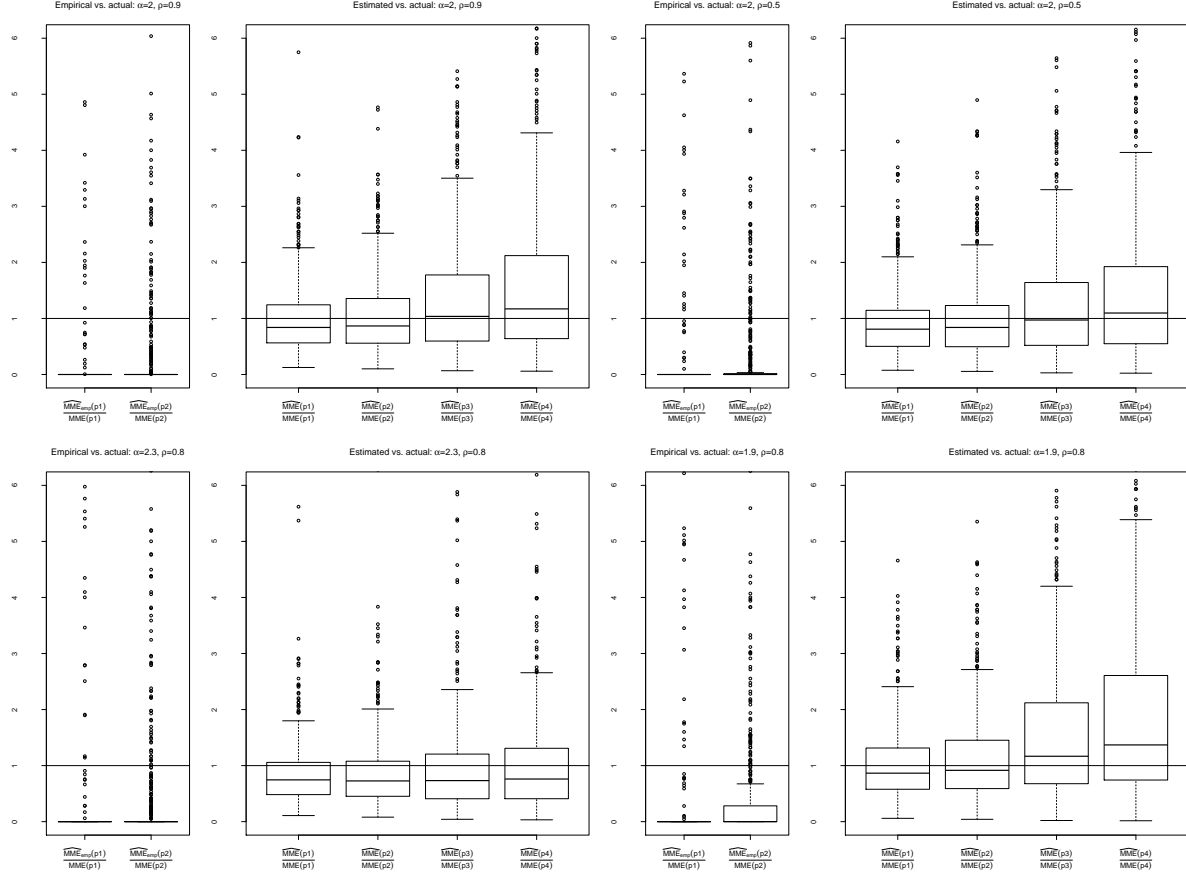


Figure 5.1. Box plots of $\widehat{\text{MME}}_{\text{emp}}(p)/\text{MME}(p)$ with $p_1 = 1/500, 1/1000$ and of $\widehat{\text{MME}}(p)/\text{MME}(p)$ with $p_1 = 1/500, p_2 = 1/1000, p_3 = 1/5000, p_4 = 1/10000$ for Example 5.3 with Gaussian copula: (a) top left: $\alpha = 2, \rho = 0.9$ and $\alpha_0 = 2.1$; (b) top right: $\alpha = 2, \rho = 0.5$ and $\alpha_0 = 2.67$; (c) bottom left: $\alpha = 2.3, \rho = 0.8$ and $\alpha_0 = 2.55$, (d) bottom right: $\alpha = 1.9, \rho = 0.8$ and $\alpha_0 = 2.11$.

In order to check the performance of the extreme value when $p \ll 1/n$ we create box-plots for $\widehat{\text{MME}}_{\text{emp}}(p)$ from 500 samples in each of the four models, where $n = 1000, k = 100$ and we restrict to 4 values of p given by $1/500, 1/1000, 1/5000, 1/10000$. The plot is given in Figure 5.1. Overall the ratio of the estimate to its real target value seem close to one, and we conclude that the estimators are reasonably good.

Example 5.4 (MARSHALL-OLKIN COPULA) Suppose (Z_1, Z_2) has identical Pareto marginal distributions with parameter $\alpha > 0$ and a dependence structure given by a *Marshall-Olkin survival copula* with parameters $\gamma_1, \gamma_2 \in (0, 1)$ as given in Remark 3.3 (d).

We note a parameter restriction from Remark 3.3 (d) is given by $\min(\gamma_1, \gamma_2) \in (1 - 1/\alpha, 1)$. Hence, we find estimates of MME for the $\gamma_1 \geq \gamma_2$ case but not for MES in this example. For $\gamma_1 \geq \gamma_2$, we can explicitly compute

$$\text{MME}(p) = \frac{1}{\alpha - 1} p^{1 - \gamma_2 - 1/\alpha}.$$

In our study we generate the above distribution for two sets of choice of parameters:

- (a) $\alpha = 2, \gamma_1 = 0.8, \gamma_2 = 0.7$. Hence $\alpha_0 = 2.6$.
- (b) $\alpha = 2.5, \gamma_1 = 0.8, \gamma_2 = 0.8$. Hence $\alpha_0 = 3$.

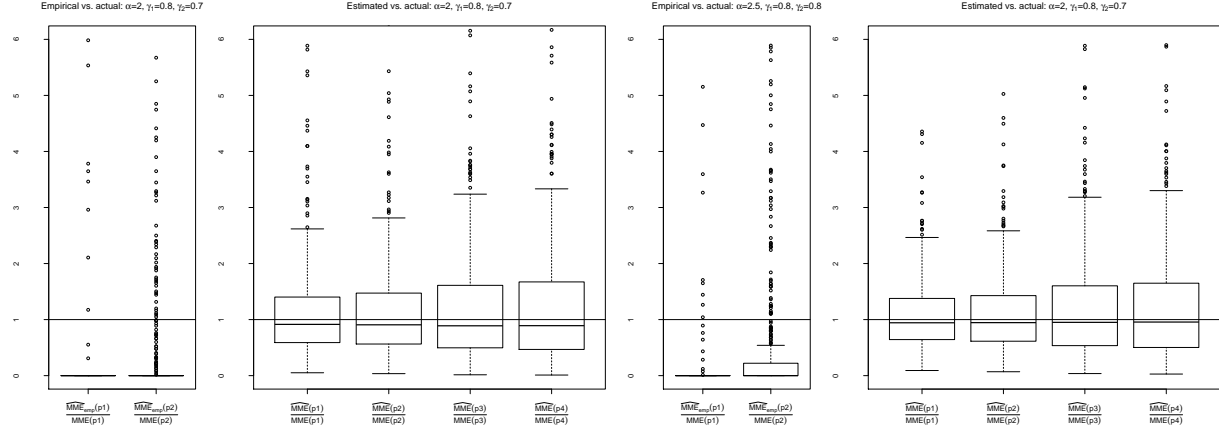


Figure 5.2. Box plots of $\widehat{MME}_{\text{emp}}(p)/MME(p)$ with $p_1 = 1/500, 1/1000$ and $\widehat{MME}(p)/MME(p)$ with $p_1 = 1/500, p_2 = 1/1000, p_3 = 1/5000, p_4 = 1/10000$ for Example 5.4 with Marshall-Olkin copula: (a) left two plots: $\alpha = 2, \gamma_1 = 0.8, \gamma_2 = 0.7$ and $\alpha_0 = 2.6$; (b) right two plots: $\alpha = 2.5, \gamma_1 = 0.8, \gamma_2 = 0.8$ and $\alpha_0 = 3$.

In Figure 5.2, we create box-plots for $\frac{\widehat{MME}}{MME}$ from 500 samples in each of the four models, where $n = 1000, k = 100$ and we restrict to 4 values of p given by $1/500, 1/1000, 1/5000, 1/10000$. Again we observe that the ratio of the estimate to its real target value seem to be close to one, and we conclude that the estimators are reasonably good.

Example 5.5 (MODEL C) We look at C where $\mathbf{Y} = (Y_1, Y_2)$ and Y_1, Y_2 are iid Pareto (α) random variables, $\mathbf{V} = (V_1, V_2)$ with $V_1 = V_2$ following Pareto (α_0) and $\mathbf{Z} = \mathbf{Y} + \mathbf{V}$. Using Theorem 2.12 we can check that $\mathbf{Z} \in \mathcal{MRV}(\alpha, b, \nu) \cap \mathcal{HRV}(\alpha_0, b_0, \nu_0)$ if $\alpha < \alpha_0 < \alpha + 1$ and all conditions (A), (B1) and (B2) are satisfied. Thus, we can find limits for both $MME(p)$ and $MES(p)$ for p going to 0. It is also possible to calculate MME and MES explicitly. We do so for $\alpha = 1.5$ and $\alpha_0 = 2$ here.

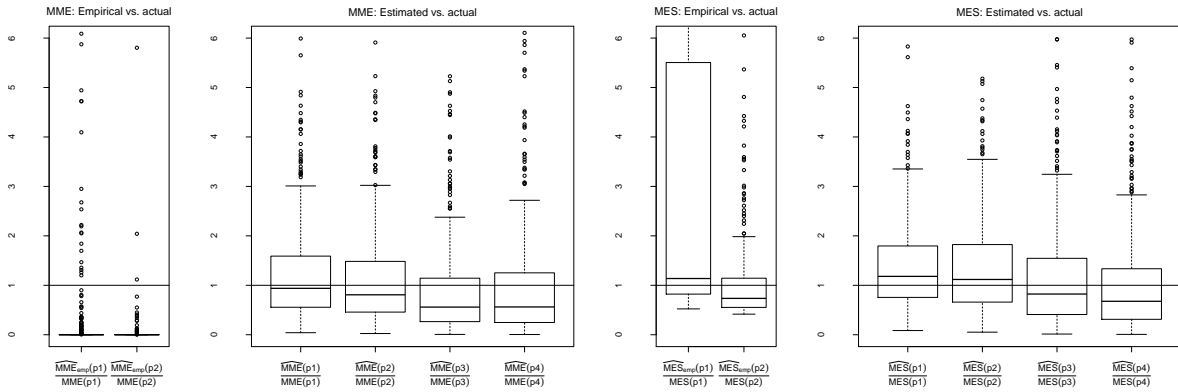


Figure 5.3. (a) Left two plots: Box plots of $\widehat{MME}_{\text{emp}}(p)/MME(p)$ with $p_1 = 1/500, p_2 = 1/1000$ as well as $\widehat{MME}(p)/MME(p)$ with $p_1 = 1/500, p_2 = 1/1000, p_3 = 1/5000, p_4 = 1/10000$ for Model C in Example 5.5 with $\alpha = 1.5, \alpha_0 = 2$. (b) Right two plots: Analog plots for MES .

We found that the Hill plots were not that stable, hence we used an L-moment estimator (or a probability weighted moment estimator) to estimate α and α_0 ; see Hosking (1990), de Haan and Ferreira (2006) for

details. The estimates of the tail parameters are not shown here. In Figure 5.3, we create box-plots for $\frac{\widehat{\text{MME}}}{\text{MME}}$ and $\frac{\widehat{\text{MES}}}{\text{MES}}$ where $n = 1000, k = 100$ with 500 samples and we restrict to 4 values of p given by $1/500, 1/1000, 1/5000, 1/10000$. The ratios of the estimators and the targets seem close to one. Of course, the empirical estimators for $p = 1/500, p = 1/1000$ do not perform so well.

5.3. Data Example: Returns from Netflix and S&P

In this section we use the method we developed in order to estimate MME and MES from a real data set. We observe a data set which exhibits *asymptotic tail independence* and we compare estimates of both statistics (MME and MES) under this assumption versus a case when we use a formula that does not assume asymptotic independence (similar to estimates obtained in Cai et al. (2015)).

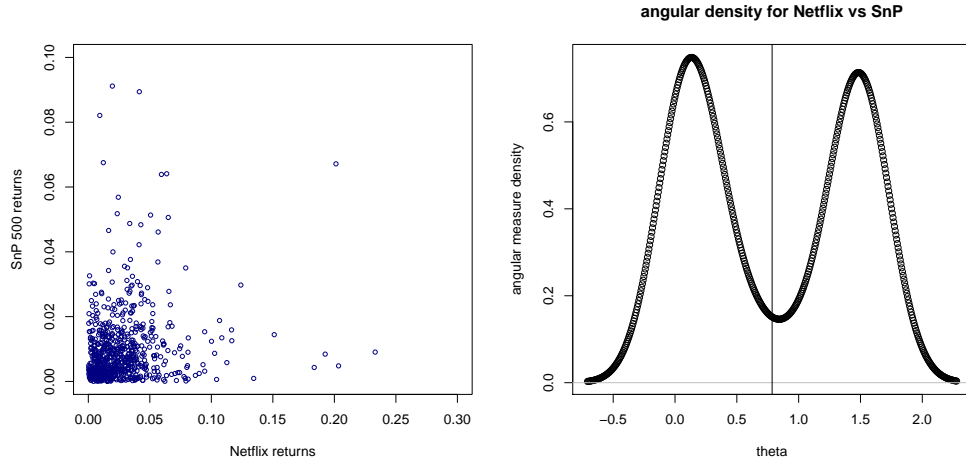


Figure 5.4. Left plot: Scatter plot of (NFLX, SNP). Right plot: angular density plot to of the rank-transformed returns data.

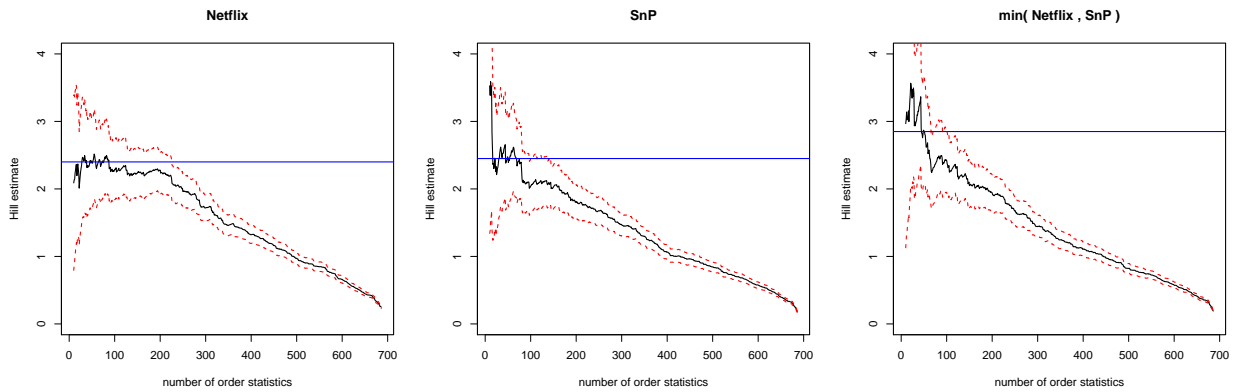


Figure 5.5. Hill plots of the tail parameters of the two negative returns (NFLX, SNP) and that of hidden tail parameter α_0 estimated using $\min(\text{NFLX}, \text{SNP})$.

We observe return values from daily equity prices of Netflix (NASDAQ:NFLX) as well as daily return

values from S&P 500 index for the period January 1, 2004 to December 31, 2013. The data was downloaded from *Yahoo Finance* (<http://finance.yahoo.com/>). The entire data set uses 2517 trading days out of which 687 days exhibited negative returns in both components and we used these 687 data points for our study.

A scatter plot of the returns data shows some concentration around the axes but the data seems to exhibit some positive dependence of the variables too; see leftmost plot in Figure 5.4. Since the scatterplot doesn't clearly show whether the data has asymptotic tail independence or not, we create an angular density plot of the rank-transformed data. Under asymptotic independence we should observe two peaks in the density, one concentrating around 0 and the other around $\pi/2$, which is what we see in the right plot in Figure 5.4; see Resnick (2007) for further discussion on the angular density. Hence, we can discern that our data exhibits asymptotic tail independence and proceed to compute the hidden regular variation tail parameter using $\min(\text{NFLX}, \text{SNP})$ as the data used to get a Hill estimate of α_0 . The left two plots in Figure 5.5 show Hill plots of both the Netflix negative returns (NFLX) and the S&P 500 negative returns (SNP). A QQ plot (not shown) suggests that both margins are heavy-tailed and by choosing $k = 50$ for the Hill-estimator we obtain as estimate of the tail parameters $\hat{\alpha}_{\text{NFLX}} = 2.39, \hat{\alpha}_{\text{SNP}} = 2.46$ (indicated by blue horizontal lines in the plot). Again using a Hill-estimator with $k = 50$, the estimate $\hat{\alpha}_0 = 2.86$ is obtained; see the rightmost plot in Figure 5.5.

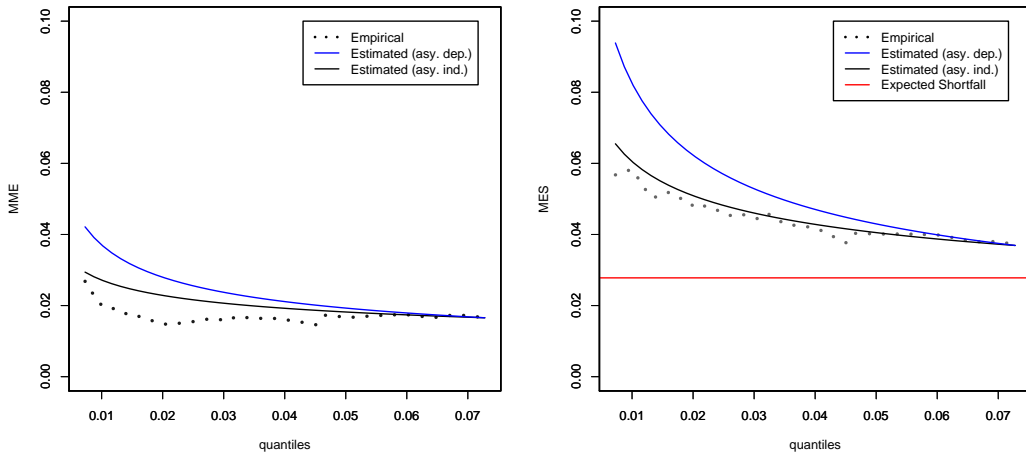


Figure 5.6. MME and MES plots under the tail dependence model as well as the asymptotic independent model.

Now, we use the values of $\hat{\alpha}_{\text{SNP}} = 2.46$ and $\hat{\alpha}_0 = 2.86$ to compute estimated values of MME and MES. In Figure 5.6 we plot the empirical estimates of MME and MES (dotted lines), the extreme value estimate without assuming asymptotic independence (blue bold line) and the extreme value estimate assuming asymptotic independence (black bold line). We observe that both MME and MES values are smaller under the assumption of asymptotic independence than in the case where we do not assume asymptotic independence. Hence, without an assumption of asymptotic independence, the firm might over-estimate its' capital shortfall if the systemic returns tend to show an extreme loss.

6. Conclusion

In this paper we study two measures of systemic risk, namely *Marginal Expected Shortfall* and another alternative we propose called *Marginal Mean Excess* in the presence of asymptotic independence of the marginal distributions in a bivariate set-up. We specifically observe that the very useful Gaussian copula model with Pareto-type tails satisfy our model assumptions for the MME and we can find the right rate of

increase (decrease) of MME in this case. Moreover we observe that, if the data we have, exhibits *asymptotic tail independence*, then we can provide an estimate of MME that is closer to the empirical estimate (and possibly smaller) than the one that would be obtained if we did not assume *asymptotic tail independence*.

In a companion paper, [Das and Fasen \(2016\)](#), we investigate various copula models and mixture models that satisfy our assumptions under which we can find asymptotic limits of MME and MES. A further direction of work would involve finding the influence of multiple system-wide risk events (for example, multiple market indicators) on a single or group of components (for example, one or more financial institutions).

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